AN EXPLANATION
OF THE OBSERVED
IRREGULARITIES IN THE
MOTION OF URANUS

JOHN C. ADAMS
AN EXPLANATION

OF THE

OBSERVED IRREGULARITIES

IN THE

MOTION OF URANUS,

ON THE HYPOTHESIS OF DISTURBANCES

CAUSED BY A MORE DISTANT PLANET;

WITH A DETERMINATION OF THE MASS, ORBIT, AND POSITION
OF THE DISTURBING BODY.

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(From the Appendix to the Nautical Almanac for the Year 1851.)

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ADVERTISEMENT.

This Paper was communicated by the Author to the Royal Astronomical Society, and was read to that body, at their ordinary meeting, on November 13, 1846. The Press of the Society being engaged on an extensive paper, on the longitude of Valentia, by the Astronomer Royal, and it being deemed of national importance that Mr. Adams's Paper should be submitted to the world without loss of time, application was made to Capt. W. H. Smyth, R.N., President, and to the Rev. R. Sheepshanks, Secretary, of the Society, who, with their usual promptitude and zeal, granted permission for the immediate printing and publishing of the Paper by the Nautical Almanac Office; and it is under these circumstances that the investigations of Mr. Adams first appear as an extract from the Appendix to the Nautical Almanac for 1851.

W. S. STRATFORD,
Superintendent of the Nautical Almanac.

Nautical Almanac Office,
December 31, 1846.
ON THE PERTURBATIONS OF URANUS.

1. The irregularities in the motions of Uranus have for a long time engaged the attention of Astronomers. When the path of the planet became approximately known, it was found that, previously to its discovery by Sir W. Herschel in 1781, it had several times been observed as a fixed star by Flamsteed, Bradley, Mayer, and Lemonnier. Although these observations are doubtless very far inferior in accuracy to the modern ones, they must be considered valuable, in consequence of the great extension which they give to the observed arc of the planet's orbit. Bouvard, however, to whom we owe the Tables of Uranus at present in use, found that it was impossible to satisfy these observations, without attributing much larger errors to the modern observations than they admit of, and consequently founded his Tables exclusively on the latter. But in a very few years sensible errors began again to show themselves, and though the Tables were formed so recently as 1821, their error at the present time exceeds two minutes of space, and is still rapidly increasing. There appeared, therefore, no longer any sufficient reason for rejecting the ancient observations, especially since, with the exception of Flamsteed's first observation, which is more than twenty years anterior to any of the others, they are mutually confirmatory of each other.

2. Now that the discovery of another planet has confirmed in the most brilliant manner the conclusions of analysis, and enabled us with certainty to refer these irregularities to their true cause, it is unnecessary for me to enter at length upon the reasons which led me to reject the various other hypotheses which had been formed to account for them. It is sufficient to say, that they all appeared to be very improbable in themselves, and incapable of being tested by any exact calculation. Some had even supposed that at the great distance of Uranus from the Sun, the law of attraction becomes different from that of the inverse square of the distance. But the law of gravitation was too firmly established for this to be admitted, till every other hypothesis had failed, and I felt convinced that in this, as in every previous instance of the kind, the discrepancies which had for a time thrown doubts on the truth of the law, would eventually afford the most striking confirmation of it.

3. My attention was first directed to this subject several years since, by reading Mr. Airy's valuable Report on the recent progress of Astronomy. I find among my
papers the following memorandum, dated July 3, 1841:—"Formed a design, in
the beginning of this week, of investigating, as soon as possible after taking my
degree, the irregularities in the motion of Uranus which are yet unaccounted for; in
order to find whether they may be attributed to the action of an undiscovered planet
beyond it, and if possible, thence to determine approximately the elements of its
orbit, &c., which would probably lead to its discovery." Accordingly, in 1843, I
attempted a first solution of the problem, assuming the orbit to be a circle, with a
radius equal to twice the mean distance of Uranus from the Sun. Some assumption
as to the mean distance was clearly necessary in the first instance, and Bode's law
appeared to render it probable that the above would not be far from the truth. This
investigation was founded exclusively on the modern observations, and the errors of
the Tables were taken from those given in the Equations of Condition of Bouvard’s
Tables as far as the year 1821, and subsequently from the observations given in the
_Astronomische Nachrichten_, and from the Cambridge and Greenwich Observations.
The result showed that a good general agreement between theory and observation
might be obtained; but the larger differences occurring in years where the observa-
tions used were deficient in number, and the Greenwich Planetary Observations being
then in process of reduction, I applied to Mr. Airy, through the kind intervention of
Professor Challis, for the observations of some years in which the agreement appeared
least satisfactory. The Astronomer Royal, in the kindest possible manner, sent me,
in February 1844, the results of all the Greenwich Observations of Uranus.

4. Meanwhile the Royal Academy of Sciences of Göttingen had proposed the Theory
of Uranus as the subject of their mathematical prize, and although the little time
which I could spare from important duties in my college prevented me from attempt-
ing the complete examination of the theory, which a competition for the prize would
have required, yet this fact, together with the possession of such a valuable series of
observations, induced me to undertake a new solution of the problem. I now took
into account the most important terms depending on the first power of the eccentricity
of the disturbing planet, retaining the same assumption as before with respect to the
mean distance. For the modern observations, the errors of the Tables were taken
exclusively from the Greenwich Observations as far as the year 1830, with the except-
tion of an observation by Bessel, in 1823; and subsequently from the Cambridge
and Greenwich Observations, and those given in various numbers of the _Astrono-
mische Nachrichten_. The errors of the Tables for the ancient Observations were
taken from those given in the Equations of Condition of Bouvard’s Tables. After
obtaining several solutions differing little from each other, by gradually taking into
account more and more terms of the series expressing the Perturbations, I communi-
cated to Professor Challis, in September 1845, the final values which I had obtained
for the mass, heliocentric longitude, and elements of the orbit of the assumed planet.
The same results, slightly corrected, I communicated in the following month to the
Astronomer Royal. The eccentricity coming out much larger than was probable,
and later observations showing that the theory founded on the first hypothesis as to
the mean distance, was still sensibly in error, I afterwards repeated my investigation,
supposing the mean distance to be about $\frac{1}{10}$th part less than before. The result,
On the Perturbations of Uranus.

which I communicated to Mr. Airy, in the beginning of September of the present year, appeared more satisfactory than my former one, the eccentricity being smaller, and the errors of theory, compared with late observations, being less, and led me to infer that the distance should be still further diminished.

5. In November, 1845, M. Le Verrier presented to the Royal Academy of Sciences at Paris, a very complete and elaborate investigation of the Theory of Uranus, as disturbed by the action of Jupiter and Saturn, in which he pointed out several small inequalities which had previously been neglected; and in June, of the present year, he followed up this investigation by a memoir, in which he attributed the residual disturbances to the action of another planet at a distance from the Sun equal to twice that of Uranus, and found a longitude for the new planet agreeing very nearly with the result which I had obtained on the same hypothesis. On the 31st of August he presented to the Academy a more complete investigation, in which he determined the mass and the elements of the orbit of the new planet, and also obtained limiting values of the mean distance and heliocentric longitude. I mention these dates merely to show that my results were arrived at independently, and previously to the publication of those of M. Le Verrier, and not with the intention of interfering with his just claims to the honours of the discovery; for there is no doubt that his researches were first published to the world, and led to the actual discovery of the planet by Dr. Galle, so that the facts stated above cannot detract, in the slightest degree, from the credit due to M. Le Verrier.

6. In order not to have an inconvenient number of equations of condition, I divided the modern observations into groups, each including a period of three years, and as Mr. Airy had shown that the error of the Tabular Radius Vector was sometimes considerable, I either selected those observations which were made near opposition, or combined the others in such a manner that the result should be nearly free from the effects of this error. From the observations of each group, the error of the Tables in heliocentric longitude was found, corresponding to the time of mean opposition in the middle year of the group. Thus were formed 21 normal errors of the Tables, corresponding to as many equidistant periods between 1780 and 1840. The error for 1780 was found by interpolating between the errors of 1781, 1782, and 1783, and those given by the Ancient Observations of 1769 and 1771, and though not entitled to the same weight as the others, cannot, I think, be liable to much uncertainty. In my last calculations, I might have used more recent observations, but in order to obtain the effect due to the change of mean distance, it was necessary that the investigation should be founded on the same elements as before, and the later observations might be used as a test of the theory.

7. In order to satisfy myself that there was no important error in Bouvard's Tables, I recomputed all the principal inequalities produced by the action of Jupiter and Saturn, and found no difference of any consequence except in the equation depending on the mean longitude of Saturn minus twice that of Uranus, the error of which had been already pointed out by Bessel. The principal equation depending on the action of Jupiter, also required correction in consequence of the increased value which has been lately obtained for the mass of that planet. The corrections to be applied to Bouvard's Tables on these accounts, are the following:
On the Perturbations of Uranus.

\[ +1'918 \sin \phi - 2\phi_{2} - 13'1'5 \]
\[ +1'085 \sin \phi - \phi_{3} \]

\( \phi, \phi_{1}, \phi_{2} \) being the mean longitudes of Jupiter, Saturn, and Uranus, respectively.

In the Reduction of the Greenwich Observations, the latter correction was already taken into account. M. Hansen having also found some new inequalities in the motion of Uranus, depending on the square of the disturbing force, I re-computed the values of these, following the same method as that given by M. Delaunay in the *Conn. des Temps* for 1845, and my results agreed very closely with his, the terms to be added to the longitude being

\[ +32'00 \sin 3\phi_{3} - 6\phi_{1} + 2\phi + 22'18'8 \]
\[ - 8'35 \sin 2\phi_{2} - 6\phi_{1} + 2\phi + 39'10'5 \]
\[ - 1'49 \sin 4\phi_{2} - 6\phi_{1} + 2\phi + 34'48'4 \]

With respect to the inequalities of higher orders neglected by Bouvard, I considered that the most important of them would be, either those of long period, or those whose period was nearly equal to that of Uranus. During three-fourths of a revolution of the planet, the effects of the former class would be nearly confounded with those arising from a change in the epoch and mean motion, and those of the latter class with the effects produced by a constant change in the eccentricity and longitude of the Perihelion. The position of the planet to be determined would therefore be little affected by these terms, and the others would probably be much smaller than those which would necessarily be neglected in a first approximation to the perturbations produced by the new planet.

8. Taking into account the several corrections above-mentioned, the residual differences between the theoretical and observed heliocentric longitudes were the following:

<table>
<thead>
<tr>
<th>Year</th>
<th>Ancient Observations</th>
<th>Modern Observations</th>
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<tbody>
<tr>
<td>1690</td>
<td>+61'2</td>
<td>1780</td>
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<tr>
<td>1712</td>
<td>+92'7</td>
<td>1783</td>
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<tr>
<td>1715</td>
<td>+73'8</td>
<td>1786</td>
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<td>1750</td>
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<tr>
<td>1753</td>
<td>–39'5</td>
<td>1792</td>
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<td>1756</td>
<td>–45'7</td>
<td>1795</td>
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<td>1761</td>
<td>–34'9</td>
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<td>1769</td>
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9. It is easily seen that the series expressing the correction of the Mean longitude in terms of the corrections applied to the elements of the orbit, is more convergent than that which gives the correction of the true longitude, and the same thing is true for the perturbations of the mean longitude, as compared with those of the true. The corrections found above were accordingly converted into corrections of mean longitude by multiplying each of them by the factor \( \frac{p}{ab} \), \( p \) being the Rad. Vector, and \( a \) and \( b \) the semi-axes of the orbit. Hence these latter corrections were found to be the following:

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<th>Modern Observations</th>
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<td>1712</td>
<td>+54'5</td>
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<td>1840</td>
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</table>

These numbers form the basis of the subsequent investigations.

10. Let \( \delta \epsilon \), \( \delta \alpha \), \( \delta \beta \), and \( \delta \varpi \) denote the corrections to be applied to the Tabular Elements of Uranus, then the correction of the mean longitude at any time \( t \) is

\[
\delta \epsilon + 2 \epsilon \delta \varpi + \delta n - \{ 2 \cos (n t + \epsilon - \varpi) + \frac{e}{2} \cos 2 (n t + \epsilon - \varpi) \} \delta \varpi
\]

\[
+ \{ 2 \sin (n t + \epsilon - \varpi) + \frac{e}{2} \sin 2 (n t + \epsilon - \varpi) \} \delta \epsilon
\]

If we include the small term \( 2 \epsilon \delta \varpi \) in the quantity \( \delta \epsilon \), this correction may be put under the following form:

\[
\delta \epsilon + t \delta n + \cos n t \delta x_1 + \sin n t \delta y_1 + \cos 2 n t \delta x_2 + \sin 2 n t \delta y_2
\]

in which expression

\[
\delta x_2 = \frac{1}{4} e \{ \cos (\epsilon - \varpi) \delta x_1 + \sin (\epsilon - \varpi) \delta y_1 \}
\]

\[
\delta y_2 = -\frac{1}{4} e \{ \sin (\epsilon - \varpi) \delta x_1 - \cos (\epsilon - \varpi) \delta y_1 \}
\]
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11. Also, adopting the notation of Pontécoulant’s “Théorie Analytique,” the perturbations of mean longitude

\[ \begin{align*}
&= \frac{m'}{2} \sum F_i \sin i (nt - n't + \varepsilon - \varepsilon') \\
&+ m' e \sum G_i \sin \{ i (nt - n't + \varepsilon - \varepsilon') - (nt + \varepsilon - \varepsilon') \} \\
&+ m' e' \sum H_i \sin \{ i (nt - n't + \varepsilon - \varepsilon') - (nt + \varepsilon - \varepsilon') \}
\end{align*} \]

Where the accented letters belong to the disturbing planet, \( i \) takes all integral values, positive and negative, except zero, and if we put \( i (n - n') = z \), the values of \( F_i, G_i \) and \( H_i \) are the following:

\[ F_i = \left\{ \frac{3 i n^4}{z^2 (z^2 - n^2)} + \frac{i n^3}{z^3 - n^3} \right\} a \Lambda_i + \frac{2 n^2}{z (z^2 - n^2)} a' \frac{d \Lambda_i}{da} \]

\[ G_i = \left\{ \frac{3}{2} \frac{i (i-1) n^4}{z (z^2 - 2n)} - \frac{i}{2} \frac{(i+1) n^3}{z^3 - n^3} + \frac{i n^2}{z (z-n)(z-2n)} \right\} a \Lambda_i \]

\[ + \left\{ \frac{3}{2} \frac{(i-1) n^4}{z (z^2 - 2n)} - \frac{1}{2} \frac{(i-1) n^3}{z^3 - n^3} + \frac{1}{2} \frac{n^2}{z (z-n)(z-2n)} - \frac{2 i n^3}{z (z^2 - n^2)} \right\} a' \frac{d \Lambda_i}{da} \]

\[ H_i = \left\{ \frac{3}{2} \frac{(i-1) (2i-1) n^4}{z (z^2 - 2n)} + \frac{1}{2} \frac{(i-1) (2i-1) n^3}{z (z^2 - 2n)} \right\} a \Lambda_{i-1} \]

\[ + \left\{ \frac{3}{2} \frac{(i-1) n^4}{z (z^2 - 2n)} + \frac{1}{2} \frac{(i-1) n^3}{z (z^2 - 2n)} + \frac{2 i n^3}{z (z-n)(z-2n)} \right\} a' \frac{d \Lambda_{i-1}}{da} \]

12. Now, if we assume \( \frac{a}{d} \) or \( \alpha = \sin 30^\circ = 0.5 \), the values of the fundamental quantities \( b, \alpha \frac{db}{d\alpha}, \alpha^2 \frac{d^2b}{d\alpha^2}, \) will be

\[ \begin{align*}
\log. b_0 &= 0 \cdot 33170; & \log. \alpha \frac{db_0}{d\alpha} &= 9 \cdot 53765 & \log. \alpha^2 \frac{d^2b_0}{d\alpha^2} &= 9 \cdot 77848 \\
\log. b_1 &= 9 \cdot 74497; & \log. \alpha \frac{db_1}{d\alpha} &= 9 \cdot 83868 & \log. \alpha^2 \frac{d^2b_1}{d\alpha^2} &= 9 \cdot 70857 \\
\log. b_2 &= 9 \cdot 32425; & \log. \alpha \frac{db_2}{d\alpha} &= 9 \cdot 68012 & \log. \alpha^2 \frac{d^2b_2}{d\alpha^2} &= 9 \cdot 87776 \\
\log. b_3 &= 8 \cdot 94670; & \log. \alpha \frac{db_3}{d\alpha} &= 9 \cdot 46315 & \log. \alpha^2 \frac{d^2b_3}{d\alpha^2} &= 9 \cdot 86253
\end{align*} \]
Hence the principal inequalities of mean longitude, produced by the action of a planet whose mass is \( \frac{m'}{5000} \), that of the Sun being unity, and the eccentricity of whose orbit is \( \frac{e'}{20} \) will be the following:

\[
-36.99 \ m' \sin \left\{ n \ t - n' t + \varepsilon - \varepsilon' \right\} \\
+58.97 \ m' \sin \left\{ 2 n \ t - n' t + \varepsilon - \varepsilon' \right\} \\
+5.80 \ m' \sin \left\{ 3 n \ t - n' t + \varepsilon - \varepsilon' \right\} \\
+2.06 \ m' \sin \left\{ n' t + \varepsilon - \omega \right\} \\
-4.30 \ m' \varepsilon' \sin \left\{ n' t + \varepsilon' - \omega' \right\} \\
+31.25 \ m' \sin \left\{ n t - 2 n' t + \varepsilon - 2 \varepsilon' + \omega \right\} \\
-12.14 \ m' \varepsilon' \sin \left\{ n t - 2 n' t + \varepsilon - 2 \varepsilon' + \omega' \right\} \\
+48.55 \ m' \sin \left\{ 2 n t - 3 n' t + 2 \varepsilon - 3 \varepsilon' + \omega \right\} \\
-39.01 \ m' \varepsilon' \sin \left\{ 2 n t - 3 n' t + 2 \varepsilon - 3 \varepsilon' + \omega' \right\}
\]

To these may be added the following, which are of two dimensions in terms of the eccentricities:

\[
+0.57 \ m' \sin 3 \left\{ n t - n' t + \varepsilon - \varepsilon' \right\} \\
-1.08 \ m' \varepsilon' \sin \left\{ 3 \left( n t - n' t + \varepsilon - \varepsilon' \right) - \omega + \omega' \right\}
\]

These expressions may be put under the following form:

\[
\begin{align*}
\lambda_1 \cos (n-n') t + \lambda_2 \cos 2(n-n') t + \lambda_3 \cos 3(n-n') t \\
+ \lambda_1 \sin (n-n') t + \lambda_2 \sin 2(n-n') t + \lambda_3 \sin 3(n-n') t \\
+ p_1 \cos n' t + p_2 \cos (n-2n') t + p_3 \cos (2n-3n') t \\
+ q_1 \sin n' t + q_2 \sin (n-2n') t + q_3 \sin (2n-3n') t
\end{align*}
\]

13. Let the time of the mean opposition in 1810 be taken as the epoch from which \( t \) is reckoned; this date, expressed in decimal parts of a year, will be 1810.328. Also, let 3 synodic periods of Uranus, \( = 3 \cdot 0362 \) years, be taken for the unit of time; then the change of the mean anomaly in an unit of time will be 13° 0' 5; also \( n=13° 0' 6, \ n'=4° 36' 0' \enspace \dot{n} = n-n'= 8° 24' 6, \ n-n'= 3° 48' 6, \ 2n-3n'=12° 13' 2 \). Hence the equations of condition given by the modern observations will be of the form

\[
c' = \delta \varepsilon + \delta x \cos \left\{ 13 \ 0' 5 \right\} + \delta \ y \cos \left\{ 26 \ 1' 0 \right\} \ t \\
+ t \delta n + \delta y_1 \sin \left\{ 13 \ 0' 5 \right\} + \delta y_2 \sin \left\{ 26 \ 1' 0 \right\} \ t \\
+ \lambda_1 \cos \left\{ 8 \ 24' 6 \right\} + \lambda_2 \cos \left\{ 16 \ 49' 2 \right\} + \lambda_3 \cos \left\{ 25 \ 13' 8 \right\} \ t \\
+ \lambda_1 \sin \left\{ 8 \ 24' 6 \right\} + \lambda_2 \sin \left\{ 16 \ 49' 2 \right\} + \lambda_3 \sin \left\{ 25 \ 13' 8 \right\} \ t \\
+ p_1 \cos \left\{ 4 \ 36' 0 \right\} + p_2 \cos \left\{ 3 \ 48' 6 \right\} + p_3 \cos \left\{ 12 \ 13' 2 \right\} \ t \\
+ q_1 \sin \left\{ 4 \ 36' 0 \right\} + q_2 \sin \left\{ 3 \ 48' 6 \right\} + q_3 \sin \left\{ 12 \ 13' 2 \right\} \ t
\]

in which \( t \) assumes all integral values from -10 to +10 in succession, and the several values of \( c' \) are contained in the table given in Article 9.
14. The final equations for the corrections of the elliptic elements will be found by multiplying each equation successively by the co-efficients of \( \delta x, \delta y, \delta x_1, \) and \( \delta y_1, \) which occur in it, and adding the several results.

Let the equations be treated in a similar manner with reference to the quantities

\[ h_1, k_1, h_2, k_2, k_3, p_3, q_3, q_5. \]

It will be seen that, in consequence of the arrangement which has been given to the equations of condition, the equations thus formed naturally separate themselves into two groups, one of which involves only \( \delta x, \delta x_1, \delta x_3, \) with the quantities \( h \) and \( p, \) while the other involves \( \delta y, \delta y_1, \delta y_3, \) with the quantities \( k \) and \( q. \)

Also, the co-efficients in these equations are easily calculated by the following formulae, putting \( t=10 \) in their right hand members:

\[
\begin{align*}
\Sigma 2 \cos m t &= \frac{\sin m (t + \frac{1}{2})}{\sin \frac{1}{2} m} \\
\Sigma 2 t \sin m t &= \frac{(t+1) \sin m t - t \sin (t+1)}{2 \sin^2 \frac{1}{2} m} \\
\Sigma 2 \cos m t \cos n t &= \frac{1}{2} \left\{ \frac{\sin (m-n) (t + \frac{1}{2})}{\sin \frac{1}{2} (m-n)} + \frac{\sin (m+n) (t + \frac{1}{2})}{\sin \frac{1}{2} (m+n)} \right\} \\
\Sigma 2 \sin m t \sin n t &= \frac{1}{2} \left\{ \frac{\sin (m-n) (t + \frac{1}{2})}{\sin \frac{1}{2} (m-n)} - \frac{\sin (m+n) (t + \frac{1}{2})}{\sin \frac{1}{2} (m+n)} \right\} \\
\Sigma 2 \cos^2 m t &= t + \frac{1}{2} + \frac{1}{2} \frac{\sin m (2 t+1)}{\sin m} \\
\Sigma 2 \sin^2 m t &= t + \frac{1}{2} - \frac{1}{2} \frac{\sin m (2 t+1)}{\sin m}
\end{align*}
\]

15. By performing the calculations, the equations of the first group are found to be the following:

\[
\begin{align*}
(c) & 151'48 = 21'0000 \delta x + 6'0670 \delta x_1 - 4'4358 \delta x_3 \\
& + 13'6320 h_1 + 0'4043 h_2 - 4'5608 h_3 \\
& + 18'6046 p_1 + 19'3384 p_2 + 7'3721 p_3 \\
(x) & 246'48 = 6'0670 \delta x + 8'2821 \delta x_1 + 4'1762 \delta x_3 \\
& + 7'4041 h_1 + 8'2523 h_2 + 4'6963 h_3 \\
& + 6'5389 p_1 + 6'3978 p_2 + 8'1831 p_3 \\
(h_1) & 209'74 = 13'6320 \delta x + 7'4041 \delta x_1 - 0'2337 \delta x_3 \\
& + 10'7022 h_1 + 4'5356 h_2 - 0'0018 h_3 \\
& + 12'7013 p_1 + 12'9883 p_2 + 8'0038 p_3 \\
(h_3) & 242'68 = 0'4043 \delta x + 8'2523 \delta x_1 + 7'5650 \delta x_3 \\
& + 4'5356 h_1 + 10'2960 h_2 + 8'1944 h_3 \\
& + 1'7866 p_1 + 1'3667 p_2 + 7'6671 p_3 \\
(h_2) & 86'67 = 4'5608 \delta x + 4'6963 \delta x_1 + 10'5023 \delta x_3 \\
& - 0'0018 h_1 + 8'1944 h_2 + 10'7071 h_3 \\
& - 3'0812 p_1 - 3'5347 p_2 + 3'8855 p_3
\end{align*}
\]
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\[(p_b) \quad 165^2 59 = 19 \cdot 3384 \cdot \delta x + 6 \cdot 3978 \cdot \delta x_1 - 3 \cdot 4948 \cdot \delta x_2 \\
+ 12 \cdot 9883 \cdot h_1 + 1 \cdot 3667 \cdot h_2 - 3 \cdot 5347 \cdot h_3 \\
+ 17 \cdot 2795 \cdot p_1 + 17 \cdot 9106 \cdot p_2 + 7 \cdot 5423 \cdot p_3 \]

\[(p_b) \quad 242 \cdot 56 = 7 \cdot 3721 \cdot \delta x + 8 \cdot 1831 \cdot \delta x_1 + 3 \cdot 4071 \cdot \delta x_2 \\
+ 8 \cdot 0038 \cdot h_1 + 7 \cdot 6671 \cdot h_2 + 3 \cdot 8855 \cdot h_3 \\
+ 7 \cdot 6127 \cdot p_1 + 7 \cdot 5423 \cdot p_2 + 8 \cdot 2019 \cdot p_3 \]

16. By means of \((x)\) eliminate \(\delta x\) from each of the other equations, and these latter become

\[(x) \quad 292^2 72 = 6 \cdot 3294 \cdot \delta x_1 + 5 \cdot 4577 \cdot \delta x_2 + 3 \cdot 4658 \cdot h_1 + 8 \cdot 1355 \cdot h_2 \\
+ 6 \cdot 0139 \cdot h_3 + 1 \cdot 1640 \cdot p_1 + 0 \cdot 8109 \cdot p_2 + 6 \cdot 0533 \cdot p_3 \]

\[(h_1) \quad 111 \cdot 41 = 3 \cdot 4658 \cdot \delta x_1 + 2 \cdot 1640 \cdot \delta x_2 + 1 \cdot 8531 \cdot h_1 + 4 \cdot 1273 \cdot h_2 \\
+ 2 \cdot 9588 \cdot h_3 + 0 \cdot 0243 \cdot p_1 + 0 \cdot 4249 \cdot p_2 + 3 \cdot 2183 \cdot p_3 \]

\[(h_2) \quad 239 \cdot 76 = 8 \cdot 1355 \cdot \delta x_1 + 7 \cdot 6504 \cdot \delta x_2 + 4 \cdot 1273 \cdot h_1 + 10 \cdot 2882 \cdot h_2 \\
+ 8 \cdot 2822 \cdot h_3 + 1 \cdot 1284 \cdot p_1 + 0 \cdot 2944 \cdot p_2 + 7 \cdot 5252 \cdot p_3 \]

\[(h_3) \quad 119 \cdot 57 = 6 \cdot 0139 \cdot \delta x_1 + 9 \cdot 3389 \cdot \delta x_2 + 9 \cdot 5888 \cdot h_1 + 8 \cdot 2822 \cdot h_2 \\
+ 9 \cdot 7166 \cdot h_3 + 0 \cdot 9593 \cdot p_1 + 0 \cdot 6652 \cdot p_2 + 5 \cdot 4866 \cdot p_3 \]

\[(p_2) \quad 26 \cdot 50 = 0 \cdot 8109 \cdot \delta x_1 + 0 \cdot 5900 \cdot \delta x_2 + 0 \cdot 4349 \cdot h_1 + 0 \cdot 9944 \cdot h_2 \\
+ 0 \cdot 6652 \cdot h_3 + 0 \cdot 1470 \cdot p_1 + 0 \cdot 1924 \cdot p_2 + 0 \cdot 7535 \cdot p_3 \]

\[(p_3) \quad 189 \cdot 38 = 6 \cdot 0533 \cdot \delta x_1 + 4 \cdot 9643 \cdot \delta x_2 + 3 \cdot 2183 \cdot h_1 + 7 \cdot 5252 \cdot h_2 \\
+ 5 \cdot 4866 \cdot h_3 + 1 \cdot 0815 \cdot p_1 + 0 \cdot 7535 \cdot p_2 + 5 \cdot 6139 \cdot p_3 \]

17. Again, by means of \((x)\) eliminate \(\delta x_1\) from each of the other equations, and we find

\[(h_1) \quad 3 \cdot 807 = -0 \cdot 2512 \cdot \delta x_2 + 0 \cdot 0135 \cdot h_1 - 0 \cdot 0452 \cdot h_2 - 0 \cdot 2334 \cdot h_3 \\
+ 0 \cdot 0065 \cdot p_1 + 0 \cdot 0045 \cdot p_2 + 0 \cdot 0052 \cdot p_3 \]

\[(h_2) \quad -12 \cdot 821 = 0 \cdot 8502 \cdot \delta x_2 - 0 \cdot 0452 \cdot h_1 + 0 \cdot 1515 \cdot h_2 + 0 \cdot 7890 \cdot h_3 \\
- 0 \cdot 0219 \cdot p_1 - 0 \cdot 0160 \cdot p_2 - 0 \cdot 0171 \cdot p_3 \]

\[(h_3) \quad -67 \cdot 149 = 4 \cdot 5120 \cdot \delta x_2 - 0 \cdot 2334 \cdot h_1 + 0 \cdot 7890 \cdot h_2 + 4 \cdot 1775 \cdot h_3 \\
- 0 \cdot 1128 \cdot p_1 - 0 \cdot 0817 \cdot p_2 - 0 \cdot 0888 \cdot p_3 \]

\[(p_1) \quad 1 \cdot 327 = -0 \cdot 0878 \cdot \delta x_2 + 0 \cdot 0045 \cdot h_1 - 0 \cdot 0160 \cdot h_2 - 0 \cdot 0817 \cdot h_3 \\
+ 0 \cdot 0024 \cdot p_1 + 0 \cdot 0017 \cdot p_2 + 0 \cdot 0018 \cdot p_3 \]

\[(p_3) \quad 1 \cdot 448 = -0 \cdot 0955 \cdot \delta x_2 + 0 \cdot 0052 \cdot h_1 - 0 \cdot 0171 \cdot h_2 - 0 \cdot 0888 \cdot h_3 \\
+ 0 \cdot 0024 \cdot p_1 + 0 \cdot 0018 \cdot p_2 + 0 \cdot 0020 \cdot p_3 \]

18. Similarly, the equations of the second group are found to be

\[(n) \quad -171^2 27 = 77 \cdot 0000 \cdot \delta n + 9 \cdot 3938 \cdot \delta y_1 - 1 \cdot 2183 \cdot \delta y_2 \\
+ 8 \cdot 8463 \cdot h_1 + 7 \cdot 3034 \cdot h_2 - 0 \cdot 5927 \cdot h_3 \\
+ 5 \cdot 7519 \cdot q_1 + 4 \cdot 8755 \cdot q_2 + 9 \cdot 5583 \cdot q_3 \]

\[(y) \quad -166 \cdot 33 = 93 \cdot 9380 \cdot \delta n + 12 \cdot 7179 \cdot \delta y_1 + 1 \cdot 8907 \cdot \delta y_2 \\
+ 11 \cdot 2022 \cdot h_1 + 11 \cdot 0849 \cdot h_2 + 2 \cdot 6731 \cdot h_3 \\
+ 7 \cdot 0936 \cdot q_1 + 5 \cdot 9913 \cdot q_2 + 12 \cdot 7441 \cdot q_3 \]
(k_1) \quad -182.87 = \quad \delta y_1 = 88.1630 \delta n + 11.2022 \delta y_1 + 0.3210 \delta y_2 \\
+ 10.2978 k_1 - 9.0964 k_2 + 0.4061 k_3 \\
+ 6.6370 q_1 + 5.6163 q_2 + 11.3316 q_3 \\

(k_2) \quad - 89.07 = \quad \delta y_1 = 73.0340 \delta n + 11.0848 \delta y_1 + 4.9266 \delta y_2 \\
+ 9.0964 k_1 + 10.7040 k_2 + 5.4376 k_3 \\
+ 5.9855 q_1 + 4.6076 q_2 + 10.9375 q_3 \\

(k_3) \quad + 124.80 = \quad \delta y_1 + 2.6731 \delta y_1 + 10.1253 \delta y_2 \\
+ 0.4061 k_1 + 5.1376 k_2 + 10.2929 k_3 \\
- 0.2197 q_1 - 0.6013 q_2 + 2.1788 q_3 \\

(q_4) \quad -107.02 = \quad \delta y_1 + 5.9913 \delta y_1 - 0.6614 \delta y_2 \\
+ 5.6163 k_1 + 4.6976 k_2 - 0.2643 k_3 \\
+ 3.6475 q_1 + 3.0894 q_2 + 6.0897 q_3 \\

(q_5) \quad -175.89 = \quad \delta y_1 + 12.7411 \delta y_1 + 1.3815 \delta y_2 \\
+ 11.3346 k_1 + 10.9375 k_2 + 2.1788 k_3 \\
+ 7.2081 q_1 + 6.0897 q_2 + 12.7981 q_3 \\

19. By means of (n), eliminate \delta n from each of the other equations, and we have

(y) \quad 42.61 = \quad \delta y_1 + 3.3771 \delta y_1 + 0.1100 k_1 + 2.1748 k_3 \\
+ 3.3962 k_3 + 0.0785 q_1 + 0.0433 q_2 + 1.0833 q_3 \\

(k_1) \quad 13.90 = \quad 0.4100 \delta y_1 + 1.0787 \delta y_1 + 0.1346 k_1 + 0.7057 k_3 \\
+ 1.0871 k_3 + 0.0288 q_1 + 0.0150 q_2 + 0.3534 q_3 \\

(k_2) \quad 73.38 = \quad 2.1748 \delta y_1 + 5.9822 \delta y_1 + 0.7057 k_1 + 3.7767 k_3 \\
+ 5.9998 k_3 + 0.1298 q_1 + 0.0732 q_2 + 1.8715 q_3 \\

(k_3) \quad 111.62 = \quad 3.3962 \delta y_1 + 10.3315 \delta y_1 + 1.0871 k_1 + 5.9998 k_3 \\
+ 10.2473 k_3 + 0.1930 q_1 + 0.1110 q_2 + 2.9145 q_3 \\

(q_4) \quad 1.42 = \quad 0.0433 \delta y_1 + 0.1100 \delta y_1 + 0.0150 k_1 + 0.0732 k_3 \\
+ 0.1110 k_3 + 0.0055 q_1 + 0.0023 q_2 + 0.0375 q_3 \\

(q_5) \quad 36.72 = \quad 1.0833 \delta y_1 + 2.8969 \delta y_1 + 0.3534 k_1 + 1.8715 k_3 \\
+ 2.9145 k_3 + 0.0684 q_1 + 0.0375 q_2 + 0.9330 q_3 \\

20. Again, eliminating \delta y_1 by means of (y) we find

(k_1) \quad 0.009 = -0.0221 \delta y_1 - 0.0010 k_1 - 0.0032 k_2 - 0.0200 k_3 \\
+ 0.0032 q_1 + 0.0009 q_2 + 0.0003 q_3 \\

(k_2) \quad -0.301 = 0.1430 \delta y_2 - 0.0032 k_1 - 0.0162 k_2 + 0.1274 k_3 \\
- 0.0059 q_1 - 0.0117 q_2 - 0.0116 q_3 \\

(k_3) \quad -3.443 = 1.2129 \delta y_2 - 0.0200 k_1 + 0.1274 k_2 + 1.0769 k_3 \\
- 0.0189 q_1 - 0.0059 q_2 - 0.0105 q_3 \\

(q_4) \quad -0.045 = -0.0062 \delta y_2 + 0.0009 k_1 - 0.0017 k_2 - 0.0059 k_3 \\
+ 0.0028 q_1 + 0.0008 q_2 + 0.0002 q_3 \\

(q_5) \quad + 0.017 = 0.0116 \delta y_2 + 0.0003 k_1 - 0.0016 k_2 - 0.0105 k_3 \\
+ 0.0008 q_1 + 0.0002 q_2 + 0.0000 q_3
21. From the equations remaining in the two groups after the elimination of \( \delta \epsilon, \delta n, \delta x_1, \delta y_1 \) it will be easy, when approximate values of the mass and mean longitude of the disturbing planet have been found, to deduce the final equations for determining these quantities more accurately by the method of minimum squares.

It may be observed, however, that the equations in each group are very nearly identical with each other, and therefore two final equations may be formed by simply adding together the several equations of each group, after giving the unknown quantities the same sign in them all. Thus we find

\[
\begin{align*}
86^\circ \cdot 552 &= -5 \cdot 7967 \delta x_1 + 0 \cdot 3018 h_1 - 1 \cdot 0188 h_2 - 5 \cdot 3704 h_3 \\
&\quad + 0 \cdot 1460 p_1 + 0 \cdot 1056 p_2 + 0 \cdot 1149 p_3 \\
3 \cdot 725 &= -1 \cdot 3958 \delta y_3 + 0 \cdot 0254 h_1 - 0 \cdot 1501 h_2 - 1 \cdot 2107 h_3 \\
&\quad + 0 \cdot 0316 q_1 + 0 \cdot 0095 q_2 + 0 \cdot 0127 q_3
\end{align*}
\]

22. If in the expressions before given for \( \delta x_6 \) and \( \delta y_6 \) we substitute \( e = 0 \cdot 046679 \) and \( \epsilon - \omega = 50^\circ 15' \cdot 8 \), we obtain

\[
\begin{align*}
\delta x_6 &= 0 \cdot 007160 \delta x_1 + 0 \cdot 009741 \delta y_1 \\
\delta y_6 &= -0 \cdot 009741 \delta x_1 + 0 \cdot 007160 \delta y_1
\end{align*}
\]

Substituting these values in the equations \((x)\) and \((y)\) and in those just found, it may be seen that by adding to the latter equations

\[
\begin{align*}
0 \cdot 006768 (x) + 0 \cdot 040287 (y)
\end{align*}
\]

and

\[
\begin{align*}
-0 \cdot 001869 (x) + 0 \cdot 008187 (y)
\end{align*}
\]

\( \delta x_1 \) and \( \delta y_1 \) will be eliminated, and we shall obtain the following equations:

\[
\begin{align*}
(1) \quad 89^\circ \cdot 641 &= 0 \cdot 3252 h_1 - 0 \cdot 9637 h_2 - 5 \cdot 3297 h_3 \\
&\quad + 0 \cdot 0165 k_1 + 0 \cdot 0876 k_2 + 0 \cdot 1368 k_3 \\
&\quad + 0 \cdot 1539 p_1 + 0 \cdot 1171 p_2 + 0 \cdot 1559 p_3 \\
&\quad + 0 \cdot 0032 q_1 + 0 \cdot 0017 q_2 + 0 \cdot 0135 q_3 \\
(2) \quad 3 \cdot 695 &= -0 \cdot 0065 h_1 - 0 \cdot 0152 h_2 - 0 \cdot 0112 h_3 \\
&\quad + 0 \cdot 0288 h_1 - 0 \cdot 1323 h_2 - 1 \cdot 2129 h_3 \\
&\quad -0 \cdot 0022 p_1 - 0 \cdot 0155 p_2 - 0 \cdot 0113 p_3 \\
&\quad + 0 \cdot 0323 q_1 + 0 \cdot 0099 q_3 + 0 \cdot 0215 q_3
\end{align*}
\]

23. These equations would be sufficient for determining the mass of the disturbing planet and its longitude at the epoch, if the eccentricity of the orbit were neglected. We will now proceed to find equations from the Ancient Observations for determining the eccentricity and longitude of the Perihelion.

The equations of condition given by the Ancient Observations are the following:

\[
\begin{align*}
62^\circ \cdot 6 &= \delta \epsilon - 0 \cdot 6776 \delta x_1 + 0 \cdot 5402 \delta x_2 + 0 \cdot 8712 h_1 + 0 \cdot 5180 h_2 \\
&\quad - 39 \cdot 31 \delta n - 0 \cdot 4795 \delta y_1 + 0 \cdot 8415 \delta y_2 + 0 \cdot 4909 h_1 + 0 \cdot 8554 h_2 \\
&\quad + 0 \cdot 0314 h_3 - 0 \cdot 9999 p_1 - 0 \cdot 8640 p_2 - 0 \cdot 5055 p_3 \\
&\quad + 0 \cdot 9995 h_5 + 0 \cdot 0145 q_1 - 0 \cdot 5035 q_2 - 0 \cdot 8628 q_3
\end{align*}
\]
### 84.5

\[
\ddot{z} + 0.4975 \: \ddot{x}_1 + 0.5050 \: \ddot{x}_2 + 0.0288 \: h_i + 0.9984 \: h_s
\]
\[
- 32.30 \: \ddot{h} + 0.8675 \: \ddot{y}_1 + 0.8631 \: \ddot{y}_2 + 0.9996 \: h_i + 0.0573 \: h_s
\]
\[
- 0.0860 \: h_i + 0.8534 \: p_1 + 0.9456 \: p_2 + 0.8220 \: p_3
\]
\[
- 0.9963 \: \dot{h}_i + 0.5213 \: q_1 + 0.8380 \: q_2 + 0.5695 \: q_3
\]

### 67.2

\[
\ddot{z} + 0.6732 \: \ddot{x}_1 - 0.0935 \: \ddot{x}_2 - 0.1120 \: h_i - 0.9749 \: h_s
\]
\[
- 31.34 \: \ddot{h} + 0.7394 \: \ddot{y}_1 + 0.9956 \: \ddot{y}_2 + 0.9937 \: h_i + 0.2297 \: h_s
\]
\[
+ 0.3305 \: h_i - 0.8105 \: p_1 + 0.9192 \: p_2 + 0.9206 \: p_3
\]
\[
- 0.9138 \: \dot{h}_i - 0.5857 \: q_1 - 0.8711 \: q_2 - 0.3905 \: q_3
\]

### 51.8

\[
\ddot{z} - 0.2616 \: \ddot{x}_1 + 0.8631 \: \ddot{x}_2 + 0.5649 \: h_i + 0.8618 \: h_s
\]
\[
- 19.59 \: \ddot{h} + 0.9652 \: \ddot{y}_1 + 0.5050 \: \ddot{y}_2 + 0.2627 \: h_i + 0.5073 \: h_s
\]
\[
- 0.6082 \: h_i + 0.0023 \: p_1 + 0.2630 \: p_2 + 0.3090 \: p_3
\]
\[
- 0.7159 \: \dot{h}_i + 0.0000 \: q_1 + 0.9642 \: q_2 + 0.8607 \: q_3
\]

### 43.2

\[
\ddot{z} - 0.4711 \: \ddot{x}_1 + 0.5505 \: \ddot{x}_2 - 0.9154 \: h_i - 0.6758 \: h_s
\]
\[
- 18.58 \: \ddot{h} + 0.8805 \: \ddot{y}_1 - 0.8348 \: \ddot{y}_2 + 0.1025 \: h_i + 0.7371 \: h_s
\]
\[
- 0.3220 \: h_i + 0.0787 \: p_1 + 0.3391 \: p_2 - 0.6814 \: p_3
\]
\[
- 0.9467 \: \dot{h}_i - 0.9969 \: q_1 - 0.9443 \: q_2 + 0.7319 \: q_3
\]

### 50.1

\[
\ddot{z} - 0.6430 \: \ddot{x}_1 - 0.1731 \: \ddot{x}_2 - 0.8543 \: h_i + 0.1599 \: h_s
\]
\[
- 17.68 \: \ddot{h} + 0.7659 \: \ddot{y}_1 - 0.9819 \: \ddot{y}_2 + 0.5198 \: h_i + 0.8879 \: h_s
\]
\[
+ 0.0686 \: h_i + 0.1510 \: p_1 + 0.3848 \: p_2 - 0.8085 \: p_3
\]
\[
- 0.9876 \: \dot{h}_i - 0.9885 \: q_1 - 0.9230 \: q_2 + 0.5885 \: q_3
\]

### 37.8

\[
\ddot{z} - 0.9492 \: \ddot{x}_1 + 0.8021 \: \ddot{x}_2 + 0.6189 \: h_i - 0.2310 \: h_s
\]
\[
- 15.25 \: \ddot{h} + 0.3145 \: \ddot{y}_1 - 0.5972 \: \ddot{y}_2 - 0.7855 \: h_i + 0.9722 \: h_s
\]
\[
+ 0.0985 \: h_3 + 0.3386 \: p_1 + 0.5287 \: p_2 - 0.9939 \: p_3
\]
\[
- 0.4179 \: \dot{h}_3 - 0.9106 \: q_1 + 0.8488 \: q_2 + 0.1100 \: q_2
\]

### 20.5

\[
\ddot{z} + 0.9955 \: \ddot{x}_1 + 0.9442 \: \ddot{x}_2 + 0.4128 \: h_i + 0.6591 \: h_s
\]
\[
- 13.60 \: \ddot{h} + 0.0538 \: \ddot{y}_1 + 0.1074 \: \ddot{y}_2 + 0.9108 \: h_i + 0.7520 \: h_s
\]
\[
+ 0.9571 \: h_3 + 0.4607 \: p_1 + 0.6182 \: p_2 - 0.9711 \: p_3
\]
\[
+ 0.2899 \: \dot{h}_3 - 0.8875 \: q_1 - 0.7860 \: q_2 - 0.2385 \: q_3
\]

### 2.4

\[
\ddot{z} - 0.9633 \: \ddot{x}_1 + 0.8560 \: \ddot{x}_2 - 0.2807 \: h_i - 0.8424 \: h_s
\]
\[
- 12.64 \: \ddot{h} + 0.2084 \: \ddot{y}_1 + 0.5170 \: \ddot{y}_2 - 0.9398 \: h_i + 0.5388 \: h_s
\]
\[
+ 0.7336 \: h_3 + 0.5279 \: p_1 + 0.6670 \: p_2 - 0.0023 \: p_3
\]
\[
+ 0.6574 \: \dot{h}_3 - 0.8493 \: q_1 + 0.7451 \: q_2 - 0.4310 \: q_3
\]

24. From each of these equations eliminate \( \ddot{x}_3 \), \( \ddot{y}_3 \), \( \ddot{z}_3 \), and \( \ddot{y}_1 \), by means of the equations \( (\ddot{z}) \), \( (\ddot{y}) \), \( (\ddot{x}) \), and \( (\ddot{y}) \) before found, and we have the following:

\[
-142.0 = 1.7265 \: \ddot{x}_2 + 0.8412 \: h_1 + 1.9521 \: h_s + 1.3230 \: h_s
\]
\[
-11.3691 \: \ddot{y}_3 + 3.6001 \: h_1 - 2.8793 \: h_s - 10.9578 \: h_s
\]
\[
-1.6779 \: p_1 + 1.6100 \: p_2 + 0.2249 \: p_3
\]
\[
+ 2.6815 \: q_1 + 1.8369 \: q_2 + 0.2995 \: q_3
\]
—105' 2 == 0 '4681  $\delta x_0 - 0 '7311 h_1 - 1 '2776 h_2 - 0 '0609 h_3
— 9 '6249  $\delta y_0 + 3 '7087 k_1 - 2 '1926 k_2 - 9 '5426 k_3
-1 '7765 p_1 - 1 '4924 p_2 + 0 '2786 p_3
+1 '6997 q_1 + 1 '1014 q_2 + 0 '7934 q_3

—126' 1 == 0 '2035  $\delta x_0 - 0 '9653 h_1 - 1 '4730 h_2 + 0 '1937 h_3
— 9 '7719  $\delta y_0 + 3 '5895 k_1 - 2 '5827 k_2 - 9 '5123 k_3
-1 '7649 p_1 - 1 '4598 p_2 + 0 '2133 p_3
+1 '5629 q_1 + 1 '0070 q_2 + 0 '8437 q_3

—199' 1 == 0 '1917  $\delta x_0 - 1 '3218 h_1 + 1 '5284 h_2 + 0 '0260 h_3
— 9 '8232  $\delta y_0 + 0 '8943 k_1 - 3 '4359 k_2 - 9 '9270 k_3
-0 '7901 p_1 - 0 '5885 p_2 - 0 '3497 p_3
+0 '2540 q_1 + 0 '1607 q_2 + 0 '4028 q_3

—174' 7 == 0 '2985  $\delta x_0 - 1 '1595 h_1 + 1 '6072 h_2 + 0 '5979 h_3
— 9 '5788  $\delta y_0 + 0 '7062 k_1 - 2 '9425 k_2 - 9 '5877 k_3
-0 '6712 p_1 - 0 '4970 p_2 - 0 '3251 p_3
+0 '1946 q_1 + 0 '1238 q_2 + 0 '3277 q_3

—166' 7 == 0 '8171  $\delta x_0 - 1 '0088 h_1 + 1 '6018 h_2 + 1 '1442 h_3
— 9 '1122  $\delta y_0 + 0 '5586 k_1 - 2 '1890 k_2 - 9 '0258 k_3
-0 '5688 p_1 - 0 '4268 p_2 - 0 '2956 p_3
+0 '1498 q_1 + 0 '0358 q_2 + 0 '2658 q_3

—114' 2 == 2 '0482  $\delta x_0 - 0 '6027 h_1 + 1 '2891 h_2 + 2 '2661 h_3
— 6 '6781  $\delta y_0 + 0 '2576 k_1 - 1 '3421 k_2 - 6 '4080 k_3
-0 '3256 p_1 - 0 '2384 p_2 - 0 '1971 p_3
+0 '0628 q_1 + 0 '0419 q_2 + 0 '1298 q_3

— 72' 4 == 2 '2815  $\delta x_0 - 0 '3786 h_1 + 0 '9257 h_2 + 2 '3601 h_3
— 4 '4181  $\delta y_0 + 0 '1283 k_1 - 0 '7339 k_2 - 4 '1495 k_3
-0 '1957 p_1 - 0 '1128 p_2 - 0 '1286 p_3
+0 '0283 q_1 + 0 '0198 q_2 + 0 '0671 q_3

— 42' 0 == 2 '1139  $\delta x_0 - 0 '2652 h_1 + 0 '6985 h_2 + 2 '1241 h_3
— 3 '1027  $\delta y_0 + 0 '0772 k_1 - 0 '4636 k_2 - 2 '8790 k_3
-0 '1348 p_1 - 0 '0984 p_2 - 0 '0924 p_3
+0 '0154 q_1 + 0 '0114 q_2 + 0 '0412 q_3

25. The largest terms depending on the eccentricity of the disturbing planet occur in $p_3$, $q_3$; it will be proper, therefore, to combine the above equations in such a manner that these quantities may acquire the largest co-efficients possible. This will be done by multiplying each equation by a quantity nearly proportional to the co-efficient of each of the unknown quantities $p_3$ and $q_3$, and adding together the several results. It was thought unsafe to employ the first of the above equations, since it is derived from the single observation of Flamsteed, made in 1690, twenty-two years anterior to any other observation.
Hence the equation for finding \( p_3 \) may be formed by multiplying the above equations, taken in order, by

\[-0.8, -0.6, +1.0, +1.0, +0.9, +0.6, +0.4, +0.3\]

beginning with the second; and the equation for \( q_3 \) by multiplying the same equations by

\[1.0, 1.0, 0.5, 0.3, 0.2, 0.1, 0.1,\]

Hence we obtain

\[
-474.1 = 4.114 \delta x_3 - 2.817 h + 7.837 k + 4.528 h
-20.745 \delta y_3 - 2.789 k - 6.551 h - 20.666 k
+0.193 p_1 + 0.377 p_2 - 1.189 p_3
-1.660 q_1 - 1.078 q_2 - 0.054 q_3
\]

\[
-485.0 = 0.446 \delta x_3 - 3.308 h - 0.142 h + 1.629 h
-32.961 \delta y_3 + 8.267 k - 8.805 k - 32.546 h
-4.173 p_1 - 3.643 p_2 + 0.037 p_3
+3.330 q_1 + 2.278 q_2 + 2.086 q_3
\]

26. Eliminate \( \delta x_3 \) and \( \delta y_3 \) from these equations by means of (x) and (y) and they become

\[
(3) - 476.7 = -2.930 h - 7.572 h + 4.332 h
-2.751 k - 6.348 k - 20.350 k
+0.155 p_1 + 0.350 p_2 - 1.686 p_3
-1.653 q_1 - 1.074 q_2 + 0.047 q_3
\]

\[
(4) - 485.9 = -3.163 h - 0.805 h + 1.360 h
+8.345 k - 8.391 k - 31.900 k
-4.525 p_1 - 3.679 p_2 - 0.233 p_3
+3.545 q_1 + 2.286 q_2 + 2.292 q_3
\]

These equations, with (1) and (2) of Article 22, suffice for the solution of our problem.

27. Eliminate the left hand members from equations (2), (3), (4), by means of equation (1) and we have

\[
0 = 0.4819 h - 0.5950 h - 5.0570 h + 0.2063 p_1 + 0.1475 p_2 + 0.1300 p_3
-0.6812 k + 3.2982 k + 29.5618 k - 0.7804 q_1 - 0.2375 q_2 - 0.4789 q_3
\]

\[
0 = -1.2005 h + 2.4466 h - 24.0122 h + 0.9735 p_1 + 0.9412 p_2 - 0.8575 p_3
-2.6633 k - 5.8825 k - 19.6219 k - 1.6362 q_1 - 1.0618 q_2 + 0.2791 q_3
\]

\[
0 = -1.7003 h - 6.0291 h - 27.5295 h - 3.6908 p_1 - 3.0772 p_2 + 0.6118 p_3
+8.4344 k - 7.9162 k - 31.1583 k + 3.5621 q_1 + 2.2954 q_2 + 2.5285 q_3
\]
28. If now we put \( \epsilon - \epsilon' = \theta \) and \( \epsilon - \varpi = \beta \), it is easily seen that

\[
\begin{align*}
\frac{h_1}{m'} &= -36\,99 \sin \theta, \\
\frac{h_2}{m'} &= -36\,99 \cos \theta, \\
\frac{h_3}{m'} &= 5\,80 \sin 3\theta, \\
\frac{h_4}{m'} &= 5\,80 \cos 3\theta, \\
\frac{p_1}{m'} &= 0 \cdot 18 \sin (\theta - \beta) - 0 \cdot 046247 \left\{ \frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta \right\} \\
\frac{q_1}{m'} &= 0 \cdot 18 \cos (\theta - \beta) + 0 \cdot 046247 \left\{ \frac{p_3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta \right\} \\
\frac{p_3}{m'} &= 24 \cdot 91 \sin (2\theta - \beta) + 0 \cdot 13055 \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} \\
\frac{q_3}{m'} &= 24 \cdot 91 \cos (2\theta - \beta) + 0 \cdot 13055 \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\}
\end{align*}
\]

29. Substituting these expressions in the equations of Art. 27, and putting for \( \beta \) its value 50° 15' 8, we obtain, after a slight reduction,

\[
\begin{align*}
0 &= -(1 \cdot 24782) \sin \theta + (1 \cdot 40248) \cos \theta - (1 \cdot 57155) \sin 2\theta + (2 \cdot 27388) \cos 2\theta \\
&- (1 \cdot 46746) \sin 3\theta + (2 \cdot 23430) \cos 3\theta + (9 \cdot 10380) \frac{p_3}{m'} - (9 \cdot 48254) \frac{q_3}{m'} \\
&+ (8 \cdot 28455) \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} - (8 \cdot 49138) \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\
&- (7 \cdot 97958) \left\{ \frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta \right\} - (8 \cdot 55742) \left\{ \frac{p_3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta \right\} \\
0 &= (1 \cdot 65083) \sin \theta + (1 \cdot 99378) \cos \theta + (2 \cdot 14259) \sin 2\theta - (2 \cdot 58192) \cos 2\theta \\
&- (2 \cdot 14400) \sin 3\theta - (2 \cdot 05631) \cos 3\theta - (9 \cdot 93475) \frac{p_3}{m'} - (8 \cdot 91803) \frac{q_3}{m'} \\
&+ (9 \cdot 08947) \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} - (9 \cdot 14306) \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\
&- (8 \cdot 65341) \left\{ \frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta \right\} - (8 \cdot 87892) \left\{ \frac{p_3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta \right\} \\
0 &= (1 \cdot 79213) \sin \theta - (2 \cdot 49403) \cos \theta - (2 \cdot 55700) \sin 2\theta - (2 \cdot 56972) \cos 2\theta \\
&- (2 \cdot 20337) \sin 3\theta - (2 \cdot 25714) \cos 3\theta + (9 \cdot 83632) \frac{p_3}{m'} + (0 \cdot 31156) \frac{q_3}{m'} \\
&- (9 \cdot 60395) \left\{ \frac{p_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} + (9 \cdot 47665) \left\{ \frac{p_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\
&+ (9 \cdot 23220) \left\{ \frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta \right\} + (9 \cdot 21679) \left\{ \frac{p_3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta \right\}
\end{align*}
\]

where the numbers enclosed within parentheses denote the logarithms of the corresponding coefficients.

30. These equations may be rapidly solved by approximation. The coefficients of $\frac{p_3}{m}$ and $\frac{q_3}{m}$ in the first equation being small, we may find from it an approximate value of $\theta$, the substitution of which in the second and third equations will give approximate values of $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$. By means of these a more accurate value of $\theta$ may be found from the first equation, and the process being repeated will enable us to satisfy all the equations as nearly as we please.

Thus we find $\theta = -51^\circ 30'$, $\frac{p_3}{m'} = 271^\circ 57'$, $\frac{q_3}{m'} = -207^\circ 24'$.

Now $\varepsilon$ is known and $= 217^\circ 55'$: $\varepsilon' = 269^\circ 25'$ the mean longitude of the disturbing planet at the epoch 1810.328. The sidereal motion in 36 synodic periods of Uranus $= 55^\circ 12'$, Precession $= 30'$, $. Mean Longitude at the time 1846.762, or October 6, 1846, $= 325^\circ 7'$.

Also, the analytical expressions for $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$ are

$$\frac{p_3}{m'} = 48.55 \sin (3 \theta - \beta) - 93.01 \varepsilon \sin (3 \theta - \beta')$$
$$\frac{q_3}{m'} = 48.55 \cos (3 \theta - \beta) - 93.01 \varepsilon \cos (3 \theta - \beta')$$

where $\varepsilon - \varepsilon' = \beta'$. Equating these to the values given above, we find $\varepsilon' = 3.2206$, $\beta' = 262^\circ 28'$, and $\beta' = 315^\circ 27'$. Hence long. of Perihelion in 1846 $= 315^\circ 57'$.

Lastly, substituting the values just obtained in equation (1), we find $m' = 0.82816$.

31. Hence the values of the mass and elements of the orbit of the disturbing planet, resulting from the first hypothesis as to the mean distance, are the following:

$$\frac{a}{a'} = 0.5$$

Mean Long. of the planet, October 6, 1846, $= 325^\circ 7'$
Longitude of the Perihelion - - - - - - 315 57
Eccentricity of the Orbit - - - - - - 0.16103
Mass (that of the Sun being 1) - - - - - - 0.0001656

These are the results which I communicated to the Astronomer Royal in October, 1845.

32. I next entered upon a similar investigation, founded on the assumption that the mean distance was about $\frac{1}{36}$th part less than before, so that $\frac{a}{a'}$ or $\alpha = \sin 31^\circ = 0.515$. The method employed was, in principle, exactly the same as that given before; but the numerical calculations were somewhat shortened by a few alterations in the process, which had been suggested by my previous solution.
33. Assuming then that \( \alpha = \sin 31^\circ \), the values of the quantities \( b, \frac{db}{d\alpha}, \frac{d^2b}{d\alpha^2} \) will be

\[
\begin{align*}
\log b_0 &= 0.33385; & \log \alpha \frac{db_0}{d\alpha} &= 9.57333; & \log \alpha^2 \frac{d^2b_0}{d\alpha^2} &= 9.82911 \\
\log b_1 &= 9.76106; & \log \alpha \frac{db_1}{d\alpha} &= 9.86149; & \log \alpha^2 \frac{d^2b_1}{d\alpha^2} &= 9.76573 \\
\log b_2 &= 9.35361; & \log \alpha \frac{db_2}{d\alpha} &= 9.71359; & \log \alpha^2 \frac{d^2b_2}{d\alpha^2} &= 9.92466 \\
\log b_3 &= 8.98918; & \log \alpha \frac{db_3}{d\alpha} &= 9.50854; & \log \alpha^2 \frac{d^2b_3}{d\alpha^2} &= 9.91563
\end{align*}
\]

Hence, by means of the formulæ given before, the principal inequalities of the mean longitude of Uranus, produced by the action of a planet whose mass is \( \frac{m'}{5000} \), that of the Sun being unity, and the eccentricity of whose orbit is \( \epsilon' = 20' \) may be found to be the following:

\[
\begin{align*}
-u &-42.33 m' \sin \{nt-n't+e-e'\} \\
+76.55 m' \sin 2 \{nt-n't+e-e'\} \\
+7.25 m' \sin 3 \{nt-n't+e-e'\} \\
+2.34 m' \sin \{n't+e'-\omega\} \\
-4.74 m' \sin' \{n't+e'-\omega'\} \\
+41.72 m' \sin \{nt-2n't+e-2e'+\omega\} \\
-16.47 m' \sin' \{nt-2n't+e-2e'+\omega'\} \\
+33.93 m' \sin \{2nt-3n't+2e-3e'+\omega\} \\
-63.11 m' \sin' \{2nt-3n't+2e-3e'+\omega'\}
\end{align*}
\]

To these we may add the following, which are of two dimensions in terms of the eccentricities:

\[
\begin{align*}
+u &+0.40 m' \sin 3 \{nt-n't+e-e'\} \\
-0.74 m' \sin' \{3(nt-n't+e-e')-\omega+\omega'\}
\end{align*}
\]

34. Now, on our present assumption, \( n = 13^0 0'\ 6', n' = 4\ 48'\ 5', n-n' = 8^0 12'\ 1, \)

\( n-2n' = 3^0 23'\ 6', 2n-3n' = 11^0 35'\ 7. \)

Hence the equations of condition given by the modern observations will be of the form

\[
\begin{align*}
c &= \delta t + \delta x_1 \cos \{13^0 0'\ 5\} t + \delta x_2 \cos \{26^0 1'\ 0\} t \\
&+ \delta n + \delta y_1 \sin \{13^0 0'\ 5\} t + \delta y_2 \sin \{26^0 1'\ 0\} t \\
&+ h_1 \cos \{8 12'\ 1\} t + h_2 \cos \{16 24'\ 2\} t + h_3 \cos \{24 36'\ 3\} t \\
&+ h_1 \sin \{8 12'\ 1\} t + h_2 \sin \{16 24'\ 2\} t + h_3 \sin \{24 36'\ 3\} t \\
&+ p_1 \cos \{4.48'\ 5\} t + p_2 \cos \{3 23'\ 6\} t + p_3 \cos \{11 35'\ 7\} t \\
&+ q_1 \sin \{4.48'\ 5\} t + q_2 \sin \{3 23'\ 6\} t + q_3 \sin \{11 35'\ 7\} t
\end{align*}
\]
35. Treating these equations of condition in the same manner as before, the equations in the first group, derived from them, are found to be the following:

\[
\begin{align*}
(e) & \quad 151'48 = 21'0000 \delta e + 6'0670 \delta x_1 - 4'1358 \delta x_2 \\
& \quad + 13'9515 \dot{h}_4 + 0'9171 \dot{h}_3 - 4'5965 \dot{h}_4 \\
& \quad + 18'3916 \dot{p}_1 + 19'6752 \dot{p}_s + 8'4184 \dot{p}_s \\
(x) & \quad 216'18 = 6'0670 \delta e + 8'2821 \delta x_1 + 4'1762 \delta x_2 \\
& \quad + 7'3510 \dot{h}_4 + 8'3027 \dot{h}_3 + 5'0961 \dot{h}_4 \\
& \quad + 6'5793 \dot{p}_1 + 6'3319 \dot{p}_s + 8'0850 \dot{p}_s \\
(h) & \quad 207'58 = 13'9515 \delta e + 7'3340 \delta x_1 - 0'4177 \delta x_2 \\
& \quad + 10'9735 \dot{h}_4 + 4'6775 \dot{h}_3 - 0'0005 \dot{h}_3 \\
& \quad + 12'8697 \dot{p}_1 + 13'4050 \dot{p}_s + 8'4781 \dot{p}_s \\
(h) & \quad 245'17 = 0'9471 \delta e + 8'3027 \delta x_1 + 7'2362 \delta x_2 \\
& \quad + 4'6775 \dot{h}_4 + 10'0259 \dot{h}_3 + 8'3220 \dot{h}_3 \\
& \quad + 2'3661 \dot{p}_1 + 1'6727 \dot{p}_s + 7'3073 \dot{p}_s \\
(h) & \quad 103'48 = 4'5965 \delta e + 5'0961 \delta x_1 + 10'5558 \delta x_2 \\
& \quad - 0'0005 \dot{h}_4 + 8'3220 \dot{h}_3 + 10'9749 \dot{h}_3 \\
& \quad - 2'8935 \dot{p}_1 - 3'7316 \dot{p}_s + 3'3852 \dot{p}_s \\
\end{align*}
\]

36. Similarly the equations in the second group, are

\[
\begin{align*}
(n) & \quad -171'27 = 77'0000 \delta n + 9'3938 \delta y_1 - 1'2183 \delta y_2 \\
& \quad + 8'7355 \dot{h}_4 + 7'6213 \dot{h}_3 - 0'0590 \dot{h}_3 \\
& \quad + 5'9764 \dot{q}_1 + 4'3975 \dot{q}_s + 9'6152 \dot{q}_s \\
(y) & \quad -166'33 = 93'9380 \delta n + 12'7179 \delta y_1 + 1'8907 \delta y_2 \\
& \quad + 11'0393 \dot{h}_4 + 11'3717 \dot{h}_3 + 3'3196 \dot{h}_3 \\
& \quad + 7'3747 \dot{q}_1 + 5'3825 \dot{q}_s + 12'6816 \dot{q}_s \\
(h) & \quad -181'31 = 87'3550 \delta n + 11'0393 \delta y_1 - 0'3758 \delta y_2 \\
& \quad + 10'0261 \dot{h}_4 + 9'2740 \dot{h}_3 + 0'9476 \dot{h}_3 \\
& \quad + 6'8054 \dot{q}_1 + 4'9866 \dot{q}_s + 11'1971 \dot{q}_s \\
(k) & \quad -99'51 = 76'2130 \delta n + 11'3717 \delta y_1 + 4'4810 \delta y_2 \\
& \quad + 9'2740 \dot{h}_4 + 10'9740 \dot{h}_3 + 5'6294 \dot{h}_3 \\
& \quad + 6'0323 \dot{q}_1 + 4'3916 \dot{q}_s + 11'0843 \dot{q}_s \\
(k) & \quad 113'14 = - 0'5900 \delta n + 3'3196 \delta y_1 + 10'2112 \delta y_2 \\
& \quad + 0'9476 \dot{h}_4 + 5'6294 \dot{h}_3 + 10'0251 \dot{h}_3 \\
& \quad + 0'1746 \dot{q}_1 + 0'4543 \dot{q}_s + 2'4791 \dot{q}_s \\
\end{align*}
\]

37. The equations \((p_1), (p_2)\) of the first group, and \((q_0), (q_0)\) of the second were not formed, as our previous solution showed that when \(\delta e, \delta n, \delta x_1\), and \(\delta y_1\), were eliminated, the co-efficients of the remaining unknown quantities in these equations would be extremely small. It will be preferable to combine the equations \((h_1), (h_2), (h_3), \) and \((k_1), (k_2), (k_3)\) before, instead of after, the elimination of \(\delta e, \delta n, \delta x_1, \) and \(\delta y_1\), from them.
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If then we change the sign of the third equation in each group, and add it to the fourth and fifth, we obtain

\[ 111'07 = -17 \cdot 6009 \delta x + 6 \cdot 0448 \delta x_1 + 18 \cdot 2907 \delta x_3 - 6 \cdot 2965 h_1 + 13 \cdot 6704 h_2 + 19 \cdot 2974 h_3 - 13 \cdot 3971 p_1 - 15 \cdot 4639 p_2 + 2 \cdot 4144 p_3 \]

\[ 194'94 = -11 \cdot 7320 \delta n + 3 \cdot 6520 \delta y_1 + 15 \cdot 0680 \delta y_2 + 0 \cdot 1951 h_1 + 7 \cdot 3294 h_2 + 14 \cdot 7069 h_3 - 0 \cdot 5785 q_1 - 0 \cdot 5496 q_2 + 2 \cdot 2363 q_3 \]

38. By means of (\(e\)) and (\(n\)) of Articles 35 and 36, eliminate \(\delta e\) and \(\delta n\) from (\(x\)) and (\(y\)), and also from the equations just found, and we have

\[
(x) \ 202'72 = \ 6 \cdot 5294 \ \delta x_1 + 5 \cdot 4577 \ \delta x_3 + 3 \cdot 3231 h_1 + 8 \cdot 0291 h_3 + 6 \cdot 2410 h_4 + 1 \cdot 2659 p_1 + 0 \cdot 6177 p_2 + 5 \cdot 6529 p_3
\]

\[
(y) \ 42'61 = \ 1 \cdot 2578 \ \delta y_1 + 3 \cdot 3771 \ \delta y_3 + 0 \cdot 3822 h_1 + 2 \cdot 0739 h_2 + 3 \cdot 3016 h_3 + 0 \cdot 0836 q_1 + 0 \cdot 0298 q_2 + 0 \cdot 9513 q_3
\]

\[
268'02 = \ 11 \cdot 1297 \ \delta x_1 + 14 \cdot 4919 \ \delta x_3 + 5 \cdot 3967 h_1 + 14 \cdot 4642 h_2 + 15 \cdot 4449 h_3 + 2 \cdot 0175 p_1 + 1 \cdot 0266 p_2 + 9 \cdot 4702 p_3
\]

\[
168'85 = \ 5 \cdot 0833 \ \delta y_1 + 14 \cdot 8824 \ \delta y_3 + 1 \cdot 5261 h_1 + 8 \cdot 4906 h_2 + 14 \cdot 6979 h_3 + 0 \cdot 3320 q_1 + 0 \cdot 1189 q_2 + 3 \cdot 3813 q_3
\]

39. Substituting for \(\delta x_5, \delta y_5\) their values in terms of \(\delta x_1, \delta y_1\), we find

\[
6 \cdot 5294 \ \delta x_1 + 5 \cdot 4577 \ \delta x_3 = \ 6 \cdot 5700 \ \delta x_1 + 0 \cdot 0490 \ \delta y_1
\]

\[
1 \cdot 2578 \ \delta y_1 + 3 \cdot 3771 \ \delta y_3 = -0 \cdot 0303 \ \delta x_1 + 1 \cdot 2829 \ \delta y_1
\]

\[
11 \cdot 1297 \ \delta x_1 + 14 \cdot 4919 \ \delta x_3 = 11 \cdot 2378 \ \delta x_1 + 0 \cdot 1300 \ \delta y_1
\]

\[
5 \cdot 0833 \ \delta y_1 + 14 \cdot 8824 \ \delta y_3 = -0 \cdot 1335 \ \delta x_1 + 5 \cdot 1943 \ \delta y_1
\]

Hence, if we add to the two latter equations

\[
-1 \cdot 7106 \ (x) - 0 \cdot 03607 \ (y)
\]

and

\[
0 \cdot 00165 \ (x) - 4 \cdot 0487 \ (y)
\]

\(\delta x_1\) and \(\delta y_1\) will be eliminated, and we shall obtain the following equations:

\[
(1) \ 80'28 = \ 0 \cdot 2883 h_1 - 0 \cdot 7295 h_3 - 4 \cdot 4559 h_9 + 0 \cdot 0138 \tilde{h}_1 + 0 \cdot 0748 h_3 + 0 \cdot 1223 h_9 + 0 \cdot 1479 \tilde{p}_1 + 0 \cdot 0813 p_2 + 0 \cdot 1997 p_9 + 0 \cdot 0030 q_1 + 0 \cdot 0111 q_2 + 0 \cdot 0343 q_9
\]

\[
(2) \ 3'34 = -0 \cdot 0055 h_1 - 0 \cdot 0132 h_3 - 0 \cdot 0106 h_9 + 0 \cdot 0212 \tilde{h}_1 - 0 \cdot 0939 h_3 - 0 \cdot 9662 h_9 - 0 \cdot 0021 \tilde{p}_1 - 0 \cdot 0011 p_2 - 0 \cdot 0093 p_9 + 0 \cdot 0066 q_1 + 0 \cdot 0017 q_2 + 0 \cdot 0203 q_9
\]
40. Again, the equations of condition given by the Ancient Observations are

\[
\begin{align*}
62 \cdot 6 &= \delta_e - 0.8776 \delta x_1 + 0.5402 \delta x_2 + 0.7923 h_1 + 0.2554 h_2 \\
-39 \cdot 31 \delta n &= -0.4793 \delta y_1 + 0.8415 \delta y_2 + 0.6101 h_1 + 0.9688 h_2 \\
-0.3575 h_3 &= 0.9777 p_1 - 0.6870 p_2 - 0.1009 p_3 \\
+0.2319 h_4 &= 0.1566 q_1 - 0.7267 q_2 - 0.9949 q_3 \\
84 \cdot 5 &= \delta e + 0.4975 \delta x_1 - 0.5050 \delta x_2 - 0.0887 h_1 - 0.9843 h_2 \\
-32 \cdot 30 \delta n &= -0.8675 \delta y_1 - 0.8631 \delta y_2 + 0.9961 h_1 - 0.1767 h_2 \\
+0.2634 h_3 &= 0.9085 p_1 - 0.3355 p_2 + 0.9681 p_3 \\
-0.9647 h_4 &= -0.7178 q_1 - 0.9420 q_2 + 0.2506 q_3 \\
67 \cdot 2 &= \delta e + 0.6732 \delta x_1 - 0.0935 \delta x_2 - 0.2243 h_1 - 0.8994 h_2 \\
-31 \cdot 34 \delta n &= -0.7394 \delta y_1 - 0.9566 \delta y_2 + 0.9745 h_1 - 0.4371 h_2 \\
+0.6277 h_3 &= -0.8720 p_1 - 0.2815 p_2 + 0.9982 p_3 \\
-0.7785 h_4 &= -0.1895 q_1 - 0.9596 q_2 - 0.0591 q_3 \\
-51 \cdot 8 &= \delta e - 0.2616 \delta x_1 - 0.8631 \delta x_2 - 0.9416 h_1 + 0.7809 h_2 \\
-19 \cdot 59 \delta n &= +0.9652 \delta y_1 - 0.5050 \delta y_2 - 0.3310 h_1 + 0.6247 h_2 \\
-0.5301 h_3 &= -0.0731 p_1 + 0.3991 p_2 - 0.6801 p_3 \\
-0.8179 h_4 &= -0.9973 q_1 - 0.9169 q_2 + 0.7331 q_3 \\
-43 \cdot 2 &= \delta e - 0.4741 \delta x_1 - 0.5505 \delta x_2 - 0.8861 h_1 + 0.5704 h_2 \\
-18 \cdot 58 \delta n &= +0.8805 \delta y_1 - 0.8348 \delta y_2 - 0.4634 h_1 + 0.8213 h_2 \\
-0.1218 h_3 &= 0.0115 p_1 + 0.1532 p_2 - 0.8147 p_3 \\
-0.9922 h_4 &= -0.9999 q_1 - 0.8914 q_2 + 0.5798 q_3 \\
-50 \cdot 1 &= \delta e - 0.6130 \delta x_1 - 0.1731 \delta x_2 - 0.8191 h_1 + 0.3120 h_2 \\
-17 \cdot 68 \delta n &= +0.7659 \delta y_1 - 0.9849 \delta y_2 - 0.5736 h_1 + 0.9397 h_2 \\
+0.2588 h_3 &= +0.0871 p_1 + 0.5001 p_2 - 0.9063 p_3 \\
-0.9659 h_4 &= -0.9962 q_1 - 0.8660 q_2 + 0.4225 q_3 \\
-37 \cdot 8 &= \delta e - 0.9492 \delta x_1 + 0.8021 \delta x_2 - 0.5743 h_1 - 0.3404 h_2 \\
-15 \cdot 25 \delta n &= +0.3145 \delta y_1 - 0.5972 \delta y_2 - 0.8186 h_1 + 0.9403 h_2 \\
+0.9652 h_3 &= +0.2872 p_1 + 0.6192 p_2 - 0.9984 p_3 \\
-0.2613 h_4 &= -0.9379 q_1 - 0.7852 q_2 + 0.0560 q_3 \\
-20 \cdot 5 &= \delta e - 0.9985 \delta x_1 + 0.9942 \delta x_2 - 0.3671 h_1 - 0.7804 h_2 \\
-13 \cdot 60 \delta n &= -0.0538 \delta y_1 + 0.1074 \delta y_2 - 0.9302 h_1 + 0.6830 h_2 \\
+0.9035 h_3 &= +0.4164 p_1 + 0.6928 p_2 - 0.9251 p_3 \\
+0.4286 h_4 &= +0.9092 q_1 + 0.7212 q_2 - 0.3756 q_3 \\
-2 \cdot 4 &= \delta e - 0.9633 \delta x_1 + 0.8560 \delta x_2 - 0.2363 h_1 - 0.8883 h_2 \\
-12 \cdot 64 \delta n &= +0.2664 \delta y_1 + 0.5170 \delta y_2 - 0.9717 h_1 + 0.4593 h_2 \\
+0.6562 h_3 &= +0.4882 p_1 + 0.7327 p_2 - 0.8345 p_3 \\
+0.7546 h_4 &= -0.8727 q_1 - 0.6806 q_2 - 0.5511 q_3
\end{align*}
\]
On the Perturbations of Uranus.

41. The equation for finding $p_\mathfrak{u}$ may be formed, as before, by multiplying the above equations taken in order by

$$-0 \cdot 8, -0 \cdot 6, +1 \cdot 0, +1 \cdot 0, +0 \cdot 9, +0 \cdot 6, +0 \cdot 4, +0 \cdot 3$$

beginning with the second; and the equation for $q_\mathfrak{u}$ by multiplying the same equations by

$$1 \cdot 0, 1 \cdot 0, 0 \cdot 5, 0 \cdot 4, 0 \cdot 3, 0 \cdot 2, 0 \cdot 1, 0 \cdot 1.$$

Thus we obtain

$$\begin{align*}
-279 \cdot 64 &= 2 \cdot 80 \ \delta x - 3 \cdot 3712 \ \delta x_1 + 0 \cdot 0265 \ \delta x_2 - 2 \cdot 9237 \ h_1 + 2 \cdot 2232 \ h_2 \\
-27 \cdot 82 \ \delta x + 3 \cdot 7593 \ \delta y_1 - 1 \cdot 0986 \ \delta y_2 - 3 \cdot 8471 \ h_1 + 3 \cdot 6706 \ h_2 \\
+0 \cdot 1281 \ h_3 + 1 \cdot 7522 \ p_1 + 2 \cdot 6081 \ p_3 - 1 \cdot 9033 \ p_3 \\
-1 \cdot 2295 \ k_2 - 3 \cdot 4661 \ q_1 - 2 \cdot 2221 \ q_2 + 1 \cdot 5785 \ q_3 \\
83 \cdot 56 &= 3 \cdot 60 \ \delta x + 0 \cdot 2714 \ \delta x_1 - 0 \cdot 9567 \ \delta x_2 - 1 \cdot 5602 \ h_1 - 1 \cdot 3924 \ h_2 \\
-91 \cdot 84 \ \delta x + 0 \cdot 5116 \ \delta y_1 - 2 \cdot 7976 \ \delta y_2 + 1 \cdot 0337 \ h_1 + 0 \cdot 6112 \ h_3 \\
+1 \cdot 0027 \ h_3 - 1 \cdot 6385 \ p_1 + 0 \cdot 1802 \ p_3 + 0 \cdot 6529 \ p_3 \\
-2 \cdot 7879 \ k_2 - 2 \cdot 4716 \ q_1 - 3 \cdot 2736 \ q_2 + 0 \cdot 3113 \ q_3
\end{align*}$$

42. Eliminate $\delta x$ and $\delta y$ by means of $(\nu)$ and $(u)$ of Articles 35 and 36, and these equations become

$$\begin{align*}
-361 \cdot 72 &= -4 \cdot 1831 \ \delta x_1 + 0 \cdot 6179 \ \delta x_2 - 4 \cdot 7839 \ h_1 + 2 \cdot 0969 \ h_2 \\
+7 \cdot 1533 \ \delta y_1 - 1 \cdot 5388 \ \delta y_2 - 0 \cdot 6999 \ h_1 + 6 \cdot 1242 \ k_2 \\
+0 \cdot 7141 \ h_3 - 0 \cdot 7000 \ p_1 - 0 \cdot 0153 \ p_3 - 6 \cdot 0258 \ p_3 \\
-1 \cdot 2508 \ h_3 - 1 \cdot 3068 \ q_1 - 0 \cdot 6369 \ q_3 + 5 \cdot 0525 \ q_3 \\
-146 \cdot 69 &= -0 \cdot 7686 \ \delta x_1 - 0 \cdot 1963 \ \delta x_2 - 3 \cdot 9519 \ h_1 - 1 \cdot 5548 \ h_2 \\
+10 \cdot 6926 \ \delta y_1 - 4 \cdot 2508 \ \delta y_2 + 11 \cdot 5128 \ h_1 + 9 \cdot 7013 \ h_2 \\
+1 \cdot 7907 \ h_3 - 1 \cdot 7913 \ p_1 - 3 \cdot 1927 \ p_3 - 0 \cdot 7902 \ p_3 \\
-2 \cdot 8583 \ h_3 + 4 \cdot 6536 \ q_1 + 1 \cdot 9555 \ q_3 + 11 \cdot 7796 \ q_3
\end{align*}$$

43. Substituting for $\delta x_2, \delta y_3$, their values in terms of $\delta x_1, \delta y_1$, we find

$$-4 \cdot 1831 \ \delta x_1 + 7 \cdot 1533 \ \delta y_1 + 0 \cdot 6179 \ \delta x_2 - 1 \cdot 5388 \ \delta y_2 = -4 \cdot 1617 \ \delta x_1 + 7 \cdot 1473 \ \delta y_1 \\
-0 \cdot 7686 \ \delta x_1 + 10 \cdot 6926 \ \delta y_1 - 0 \cdot 1963 \ \delta x_2 - 4 \cdot 2508 \ \delta y_2 = -0 \cdot 7319 \ \delta x_1 + 10 \cdot 6591 \ \delta y_1$$

Hence, if to the equations just found, we add

$$+0 \cdot 0608 \ (x) - 5 \cdot 5942 \ (y)$$

and

$$+0 \cdot 07306 \ (x) - 8 \cdot 3110 \ (y)$$

respectively,

$\delta x_1$ and $\delta y_1$ will be eliminated, and we shall obtain the following equations:

$$\begin{align*}
(3)\quad -476 \cdot 81 &= -2 \cdot 7630 \ h_1 + 6 \cdot 9793 \ h_2 + 4 \cdot 6473 \ k_3 \\
-2 \cdot 8920 \ h_1 - 5 \cdot 1777 \ h_2 - 20 \cdot 2242 \ h_3 \\
+0 \cdot 0698 \ p_1 + 0 \cdot 3785 \ p_3 - 2 \cdot 5884 \ p_3 \\
-1 \cdot 7748 \ q_1 - 0 \cdot 8036 \ q_3 - 0 \cdot 2693 \ q_3
\end{align*}$$
On the Perturbations of Uranus.

(4) \(-486^{\prime\prime} 01 = 3^{7091} h_1 -0^{9682} h_2 + 2^{2600} h_3 +8^{3364} h_1 -7^{5348} h_2 -31^{0457} h_3 -4^{4698} p_1 -3^{1451} p_2 - 0^{3772} p_3 +3^{9064} q_1 +1^{7118} q_2 + 3^{8573} q_3\)

44. Eliminate the left hand members from equations (2), (3), and (4) of Articles 39 and 43, by means of equation (1), and we have

\[0 = 0^{4200} h_1 -0^{4114} h_3 - 4^{2014} h_3 + 0^{1980} p_1 + 0^{1069} p_2 + 0^{4236} p_3 - 0^{1964} h_1 + 2^{3006} h_2 + 23^{3213} h_3 - 0^{1567} q_1 - 0^{0409} q_2 - 0^{1531} q_3\]

\[0 = -1^{0507} h_1 + 2^{6465} h_3 - 21^{8182} h_3 + 0^{9182} p_1 + 0^{8014} p_2 - 1^{4023} p_3 - 2^{1471} h_1 - 4^{1334} h_3 - 19^{4976} h_3 - 1^{7569} q_1 - 0^{7972} q_2 - 0^{0655} q_3\]

\[0 = -1^{9638} h_1 - 5^{3815} h_3 - 21^{7155} h_3 - 3^{8034} p_1 - 2^{6532} p_2 + 0^{8317} p_3 + 8^{4199} h_1 - 7^{0819} h_2 - 30^{0351} h_3 + 3^{7967} q_1 + 1^{7183} q_2 + 4^{0811} q_3\]

45. If, as before, we put \(\varepsilon - \varepsilon' = \theta\) and \(\varepsilon - \omega = \beta\), it may be seen that

\[\frac{h_1'}{m} = -4^{233} \sin \theta, \quad \frac{h_2'}{m'} = 7^{655} \sin 2 \theta\]

\[\frac{h_3'}{m} = 7^{23} \sin 3 \theta + 0^{007460} p_3 + 0^{008974} q_3\]

\[\frac{h_3}{m} = 7^{25} \cos 3 \theta - 0^{008974} p_3 + 0^{007460} q_3\]

\[\frac{p_1}{m} = 0^{20} \sin (\theta - \beta) - 0^{074738} \left\{ \frac{p_3}{m} \cos 2 \theta - \frac{q_3}{m} \sin 2 \theta \right\}\]

\[\frac{q_1}{m} = -0^{20} \cos (\theta - \beta) + 0^{074738} \left\{ \frac{p_3}{m} \sin 2 \theta + \frac{q_3}{m} \cos 2 \theta \right\}\]

\[\frac{p_3}{m} = 32^{91} \sin (2 \theta - \beta) + 0^{259765} \left\{ \frac{p_3}{m} \cos \theta - \frac{q_3}{m} \sin \theta \right\}\]

\[\frac{q_3}{m} = 32^{91} \cos (2 \theta - \beta) + 0^{259765} \left\{ \frac{p_3}{m} \sin \theta + \frac{q_3}{m} \cos \theta \right\}\]

46. Substituting these expressions in the above equations, and putting for \(\beta\) its value 50° 15' 8, we obtain

\[0 = -(1^{24872}) \sin \theta + (1^{32231}) \cos \theta - (1^{48110}) \sin 2 \theta + (2^{24256}) \cos 2 \theta - (1^{48373}) \sin 3 \theta + (2^{22809}) \cos 3 \theta + (9^{26254}) \frac{p_3}{m'} - (9^{50079}) \frac{q_3}{m'} + (8^{44376}) \left\{ \frac{p_3}{m} \cos \theta - \frac{q_3}{m} \sin \theta \right\} - (8^{02630}) \left\{ \frac{p_3}{m} \sin \theta + \frac{q_3}{m} \cos \theta \right\} - (8^{17031}) \left\{ \frac{p_3}{m} \cos 2 \theta - \frac{q_3}{m} \sin 2 \theta \right\} - (8^{06861}) \left\{ \frac{p_3}{m} \sin 2 \theta + \frac{q_3}{m} \cos 2 \theta \right\}\]
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0 = \( (1 \cdot 65190) \sin \theta + (2 \cdot 06584) \cos \theta + (2 \cdot 30220) \sin 2 \theta - (2 \cdot 60306) \cos 2 \theta \\
- (2 \cdot 19916) \sin 3 \theta - (2 \cdot 15032) \cos 3 \theta - (0 \cdot 14305) \frac{P_3}{m'} - (9 \cdot 60933) \frac{q_3}{m'} \\
+ (9 \cdot 34981) \left\{ \frac{P_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} - (9 \cdot 31615) \left\{ \frac{P_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\
- (8 \cdot 85046) \left\{ \frac{P_3}{m'} \cos 2 \theta - \frac{q_3}{m'} \sin 2 \theta \right\} - (9 \cdot 11828) \left\{ \frac{P_3}{m'} \sin 2 \theta + \frac{q_3}{m'} \cos 2 \theta \right\} \\
0 = \( (1 \cdot 91407) \sin \theta - (2 \cdot 55189) \cos \theta - (2 \cdot 62790) \sin 2 \theta \\
- (2 \cdot 25331) \sin 3 \theta - (2 \cdot 34185) \cos 3 \theta + (9 \cdot 96344) \frac{P_3}{m'} + (0 \cdot 56029) \frac{q_3}{m'} \\
- (9 \cdot 83835) \left\{ \frac{P_3}{m'} \cos \theta - \frac{q_3}{m'} \sin \theta \right\} + (9 \cdot 64968) \left\{ \frac{P_3}{m'} \sin \theta + \frac{q_3}{m'} \cos \theta \right\} \\
+ (9 \cdot 45371) \left\{ \frac{P_3}{m'} \cos 2 \theta - \frac{q_3}{m'} \sin 2 \theta \right\} + (9 \cdot 47306) \left\{ \frac{P_3}{m'} \sin 2 \theta + \frac{q_3}{m'} \cos 2 \theta \right\} \\
where the numbers enclosed within parentheses denote the logarithms of the corresponding co-efficients, as before.

47. From these equations, we find, by the same method as before

\( \theta = -46^\circ 55', \quad \frac{P_3}{m'} = 138'' 92, \quad \frac{q_3}{m'} = -109'' 83 \)

Hence, since \( \varepsilon = 217^\circ 55', \quad \varepsilon' = 264^\circ 50', \) the mean longitude of the disturbing planet at the epoch 1810 '328. The sidereal motion in 36 synodic periods of Uranus = 57° 42', Precession = 30'. Hence, Mean Longitude at the time 1846 '762, or October 6, 1846, = 323° 2'.

Also, the expressions for \( \frac{P_3}{m'} \) and \( \frac{q_3}{m'} \) are

\( \frac{P_3}{m'} = 33' 93 \sin (3 \theta - \beta) - 63' 41 \varepsilon \sin (3 \theta - \beta') \)
\( \frac{q_3}{m'} = 33' 93 \cos (3 \theta - \beta) - 63' 41 \varepsilon \cos (3 \theta - \beta') \)

where \( \varepsilon - \beta' = \beta' \).

Equating these to the values given above, we find \( \varepsilon' = 2' 4123, \beta' = 279^\circ 14', \) and \( \varepsilon' = 298^\circ 41'. \) Hence longitude of the perihelion in 1846 = 299° 11'.

Lastly, substituting the values just obtained in equation (1) of Article 39, we find \( m' = 0' 75017 \).

48. Hence the values of the mass and elements of the orbit of the disturbing planet, resulting from the second hypothesis as to the mean distance, are the following:

\( \frac{a}{\alpha} = 0' 515 \)

Mean Longitude of the Planet, October 6, 1846, = 323° 2'
Longitude of the Perihelion = 299° 11'
Eccentricity of the Orbit = 0' 120615
Mass (that of the Sun being 1) = 0' 00015003
49. From the values of $m', \theta, \frac{P_3}{m_1}$ and $\frac{q_3}{m_1}$ found above, the values of the quantities $h, k, p$, and $q$, corresponding to each hypothesis, are immediately determined. Thus we find

<table>
<thead>
<tr>
<th>1st Hypothesis</th>
<th>2nd Hypothesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{a}{d} = 0.5$</td>
<td>$\frac{a}{d} = 0.515$</td>
</tr>
</tbody>
</table>

$h_1 = 23.98, \quad k_1 = -19.07, \quad h_1 = 23.19, \quad k_1 = -21.69$

$h_2 = -47.58, \quad k_2 = -11.00, \quad h_2 = -57.30, \quad k_2 = -3.83$

$h_3 = 1.93, \quad k_3 = -7.64, \quad h_3 = -3.40, \quad k_3 = -5.76$

$p_1 = 9.93, \quad q_1 = -8.31, \quad p_1 = 6.52, \quad q_1 = -7.34$

$p_2 = -8.54, \quad q_2 = -55.36, \quad p_2 = -11.62, \quad q_2 = -54.39$

$p_3 = 224.90, \quad q_3 = -171.63, \quad p_3 = 104.21, \quad q_3 = -82.39$

50. And by substituting these values in the equations $(\varepsilon), (n), (x)$, and $(y)$, we obtain

<table>
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<tr>
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</thead>
<tbody>
<tr>
<td>$\frac{a}{d} = 0.5$</td>
<td>$\frac{a}{d} = 0.515$</td>
</tr>
</tbody>
</table>

$\delta \varepsilon = -49.77, \quad \delta n = -0.702, \quad \delta \varepsilon = -43.23, \quad \delta n = -0.5417$

$\delta x_1 = -130.69, \quad \delta y_1 = 222.38, \quad \delta x_1 = 1.77, \quad \delta y_1 = 123.98$

$\delta x_2 = 1.02, \quad \delta y_2 = 2.83, \quad \delta x_2 = 1.13, \quad \delta y_2 = 0.91$

and the corresponding corrections of the elliptic elements will be

\[
\frac{\delta a}{a} = 0.00009999, \quad \frac{\delta a}{a} = 0.00000771
\]

\[
\delta e = 20.83, \quad \delta e = 40.31
\]

\[
e \delta \omega = 127.27, \quad e \delta \omega = 47.10
\]

It will be seen that the corrections of the eccentricity and longitude of perihelion vary very rapidly with a change in the assumed mean distance.

51. If these quantities be substituted in the expressions before given, we obtain the following theoretical corrections of the mean longitude, each of these corrections being divided into two parts, of which the first is due to the changes in the elements of the orbit of Uranus, and the second to the action of the disturbing planet.
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Hypothesis I.

<table>
<thead>
<tr>
<th>Year</th>
<th>Ancient Observations.</th>
<th>Modern Observations.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>1712</td>
<td>$-288 \cdot 0 + 365 \cdot 8 = + 77 \cdot 8$</td>
<td>$-126 \cdot 12 + 129 \cdot 27 = + 3 \cdot 15$</td>
</tr>
<tr>
<td>1715</td>
<td>$-283 \cdot 1 + 357 \cdot 1 = + 74 \cdot 0$</td>
<td>$-180 \cdot 23 + 188 \cdot 70 = + 8 \cdot 42$</td>
</tr>
<tr>
<td>1750</td>
<td>$+ 210 \cdot 5 - 260 \cdot 7 = - 50 \cdot 2$</td>
<td>$-227 \cdot 66 + 210 \cdot 36 = + 12 \cdot 70$</td>
</tr>
<tr>
<td>1753</td>
<td>$+ 218 \cdot 1 - 267 \cdot 0 = - 48 \cdot 9$</td>
<td>$-265 \cdot 70 + 281 \cdot 63 = + 15 \cdot 93$</td>
</tr>
<tr>
<td>1756</td>
<td>$+ 214 \cdot 0 - 260 \cdot 0 = - 16 \cdot 0$</td>
<td>$-292 \cdot 25 + 310 \cdot 38 = + 18 \cdot 13$</td>
</tr>
<tr>
<td>1764</td>
<td>$+ 154 \cdot 0 - 186 \cdot 7 = - 32 \cdot 7$</td>
<td>$-305 \cdot 84 + 325 \cdot 27 = + 19 \cdot 43$</td>
</tr>
<tr>
<td>1769</td>
<td>$+ 79 \cdot 6 - 100 \cdot 7 = - 21 \cdot 1$</td>
<td>$-305 \cdot 67 + 325 \cdot 78 = + 20 \cdot 05$</td>
</tr>
<tr>
<td>1771</td>
<td>$+ 27 \cdot 6 - 41 \cdot 8 = - 14 \cdot 2$</td>
<td>$-291 \cdot 77 + 312 \cdot 05 = + 20 \cdot 28$</td>
</tr>
</tbody>
</table>

Hypothesis II.

<table>
<thead>
<tr>
<th>Year</th>
<th>Ancient Observations.</th>
<th>Modern Observations.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>1712</td>
<td>$-133 \cdot 7 + 211 \cdot 9 = + 78 \cdot 2$</td>
<td>$-133 \cdot 10 + 135 \cdot 98 = + 2 \cdot 88$</td>
</tr>
<tr>
<td>1715</td>
<td>$-117 \cdot 7 + 191 \cdot 5 = + 73 \cdot 8$</td>
<td>$-149 \cdot 47 + 157 \cdot 87 = + 8 \cdot 40$</td>
</tr>
<tr>
<td>1750</td>
<td>$+ 85 \cdot 2 - 134 \cdot 4 = - 49 \cdot 2$</td>
<td>$-160 \cdot 15 + 172 \cdot 99 = + 12 \cdot 84$</td>
</tr>
<tr>
<td>1753</td>
<td>$+ 73 \cdot 8 - 122 \cdot 2 = - 48 \cdot 4$</td>
<td>$-164 \cdot 52 + 180 \cdot 61 = + 16 \cdot 12$</td>
</tr>
<tr>
<td>1756</td>
<td>$+ 59 \cdot 1 - 105 \cdot 2 = - 46 \cdot 1$</td>
<td>$-162 \cdot 30 + 180 \cdot 58 = + 18 \cdot 28$</td>
</tr>
<tr>
<td>1764</td>
<td>$+ 2 \cdot 7 - 36 \cdot 4 = - 33 \cdot 7$</td>
<td>$-153 \cdot 50 + 173 \cdot 07 = + 19 \cdot 48$</td>
</tr>
<tr>
<td>1769</td>
<td>$- 43 \cdot 1 + 20 \cdot 8 = - 22 \cdot 3$</td>
<td>$-138 \cdot 87 + 158 \cdot 86 = + 19 \cdot 99$</td>
</tr>
<tr>
<td>1771</td>
<td>$- 69 \cdot 9 + 54 \cdot 7 = - 15 \cdot 2$</td>
<td>$-118 \cdot 95 + 139 \cdot 08 = + 20 \cdot 13$</td>
</tr>
</tbody>
</table>

Note: The table continues with similar entries for the years 1780 to 1840.
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52. Comparing these with the corrections of mean longitude derived from observation, we find the remaining differences to be the following:

<table>
<thead>
<tr>
<th>Year</th>
<th>Ancient Observations</th>
<th>Modern Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hypoth. I</td>
<td>Hypoth. II</td>
</tr>
<tr>
<td>1712</td>
<td>+ 6.7</td>
<td>+ 6.3</td>
</tr>
<tr>
<td>1715</td>
<td>- 6.8</td>
<td>- 6.6</td>
</tr>
<tr>
<td>1750</td>
<td>- 1.6</td>
<td>- 2.6</td>
</tr>
<tr>
<td>1753</td>
<td>+ 5.7</td>
<td>+ 5.2</td>
</tr>
<tr>
<td>1756</td>
<td>- 4.1</td>
<td>- 4.0</td>
</tr>
<tr>
<td>1764</td>
<td>- 5.1</td>
<td>- 4.1</td>
</tr>
<tr>
<td>1769</td>
<td>+ 0.6</td>
<td>+ 1.8</td>
</tr>
<tr>
<td>1771</td>
<td>+11.8</td>
<td>+12.8</td>
</tr>
</tbody>
</table>

The largest difference in the above table, viz., that for 1771, is deduced from a single observation; whereas the difference immediately preceding it, which is deduced from the mean of several, is very small.

53. The results of the two theories agree very closely with each other, and with observation, till we come to the later years of the series; and it is to be observed that the difference between the theories becomes sensible at precisely the point where they both show symptoms of diverging from the observations, the errors of the second hypothesis, however, being less than those of the other.

Recent observations show that the errors of the theory soon become very sensible, though decidedly less for the second hypothesis than for the first. The following are the differences of mean longitude as deduced from theory and observation, for the oppositions of 1843, 1844, and 1845:

<table>
<thead>
<tr>
<th>Year</th>
<th>Hypoth. I</th>
<th>Hypoth. II</th>
</tr>
</thead>
<tbody>
<tr>
<td>1843</td>
<td>+ 7.11</td>
<td>+ 5.77</td>
</tr>
<tr>
<td>1844</td>
<td>+ 8.79</td>
<td>+ 7.05</td>
</tr>
<tr>
<td>1845</td>
<td>+12.40</td>
<td>+10.18</td>
</tr>
</tbody>
</table>
For the observations of the last two years, I am indebted to the kindness of the Astronomer Royal. The three years nearly agree in showing that the errors of the first hypothesis are to those of the second in the ratio of 5 to 4, from which I inferred, in a letter to the Astronomer Royal, dated September 2, 1846, that the assumption of \( \frac{a}{a'} = \sin 35^\circ = 0.574 \), would probably satisfy all the observations very nearly.

54. The results which I have deduced from Professor Challis’s observations of the planet, strongly confirm the inference that the mean distance should be considerably diminished. It is of course impossible to determine precisely, without actual calculation, the alteration in longitude which would be produced by such a diminution in the distance. By comparing the values of \( \theta \) given by the two hypotheses, it may be seen, however, that if we took successively smaller and smaller values for the mean distance, the values found for the mean longitude in 1810 would probably go on diminishing, while at the same time the mean motion from 1810 to 1846 would rapidly increase, so that the corresponding values of the mean longitude at the present time would probably soon arrive at a minimum, and afterwards begin again to increase. This I believe to be the reason why the longitude found on the supposition of too large a value for the mean distance agrees so nearly with observation. In consequence of not making sufficient allowance for the increase in the mean motion, I hastily inferred, in my letter to the Astronomer Royal mentioned above, that the effect of a diminution in the mean distance would be to diminish the mean longitude.

55. I have already mentioned that I thought it unsafe to employ Flamsteed’s observation of 1690 in forming the equations of condition, as the interval between it and all the others is so large. The difference between it and the theory appears to be very considerable, and greater for the second hypothesis than for the first, the errors being \(+44''\cdot5\) and \(+50''\cdot0\) respectively. These errors would probably be increased by diminishing the mean distance. It would be desirable that Flamsteed’s manuscripts should be examined with reference to this point.

56. The corrections of the Tabular Radius Vector of Uranus may be easily deduced from those of the mean longitude by means of the following formula:

\[
\frac{\delta r}{r} = \frac{1}{r} \frac{d r}{d \epsilon} \frac{d \zeta - \frac{1}{2} n \frac{d}{d t} \delta \zeta + \frac{1}{4} \frac{\delta a}{a} - \frac{1}{2} \frac{e \delta e}{1 - e^2} - \frac{1}{6} m' a^2 \frac{d \Lambda'}{d a}} + \frac{m'}{2} \sum C_i \cos \{nt-n't+\epsilon-\epsilon'\}
\]

\[
+ m' e \sum D_i \cos \{i (nt-n't+\epsilon-\epsilon')-nt-\epsilon+\omega\}
\]

\[
+ m' e' \sum E_i \cos \{i (nt-n't+\epsilon-\epsilon')-nt-\epsilon+\omega'\}
\]

where \( \delta \zeta \) denotes the whole correction of the mean longitude at the time \( t \),

\[
\frac{1}{r} \frac{d r}{d \epsilon} = e \sin \{nt+\epsilon-\omega\} + \frac{3 e^2}{2} \sin 2 \{nt+\epsilon-\omega\} \text{ nearly},
\]

\[
C_i = \frac{1}{2} \frac{n}{n'-n'} a \Lambda_i
\]
On the Perturbations of Uranus.

\[ D_i = -\frac{1}{4} \frac{i n}{i \left(n-n'\right)-n} \left\{ 2 i a A_i + a^4 \frac{d A_i}{da} \right\} \]

\[ E_i = \frac{1}{4} \frac{(i-1) n}{i \left(n-n'\right)-n} \left\{ (2 i-1) a A_i + a^4 \frac{d A_i}{da} \right\} \]

\( i \) assuming all integral values positive and negative not including zero.

57. By substituting in this formula the values of \( m', \delta a, \delta e, \& c. \), already obtained, and putting \( a = 19 \, '191 \), we find the following results corresponding to the two assumed values of the mean distance.

**Hypothesis I.**

\[
\frac{a}{r} \cdot \delta r = \frac{a \, dr}{r \, d\varepsilon} \cdot \delta \zeta - \frac{a \, d\zeta}{2 \, ndt} = 0 \cdot 000089
\]

\[
+ 0 \cdot 000069 \cos \{nt-n't+\varepsilon-\varepsilon'\}
\]

\[
+ 0 \cdot 000259 \cos 2 \{nt-n't+\varepsilon-\varepsilon'\}
\]

\[
+ 0 \cdot 000109 \cos 3 \{nt-n't+\varepsilon-\varepsilon'\}
\]

\[
+ 0 \cdot 000016 \cos \{n't+\varepsilon-\varepsilon'\}
\]

\[
- 0 \cdot 000168 \cos \{nt-2n't+\varepsilon-2\varepsilon'+\varepsilon\}
\]

\[
+ 0 \cdot 000078 \cos \{nt-2n't+\varepsilon-2\varepsilon'+\varepsilon'\}
\]

\[
- 0 \cdot 000019 \cos \{2nt-3n't+2\varepsilon-3\varepsilon'+\varepsilon\}
\]

\[
+ 0 \cdot 000209 \cos \{2nt-3n't+2\varepsilon-3\varepsilon'+\varepsilon'\}
\]

**Hypothesis II.**

\[
\frac{a}{r} \cdot \delta r = \frac{a \, dr}{r \, d\varepsilon} \cdot \delta \zeta - \frac{a \, d\zeta}{2 \, ndt} = 0 \cdot 00144
\]

\[
+ 0 \cdot 000073 \cos \{nt-n't+\varepsilon-\varepsilon'\}
\]

\[
+ 0 \cdot 000266 \cos 2 \{nt-n't+\varepsilon-\varepsilon'\}
\]

\[
+ 0 \cdot 000115 \cos 3 \{nt-n't+\varepsilon-\varepsilon'\}
\]

\[
+ 0 \cdot 000016 \cos \{n't+\varepsilon-\varepsilon'\}
\]

\[
- 0 \cdot 000188 \cos \{nt-2n't+\varepsilon-2\varepsilon'+\varepsilon\}
\]

\[
+ 0 \cdot 000068 \cos \{nt-2n't+\varepsilon-2\varepsilon'+\varepsilon'\}
\]

\[
- 0 \cdot 000053 \cos \{2nt-3n't+2\varepsilon-3\varepsilon'+\varepsilon\}
\]

\[
+ 0 \cdot 000165 \cos \{2nt-3n't+2\varepsilon-3\varepsilon'+\varepsilon'\}
\]

58. The values of \( \delta \zeta \) and \( \frac{d\zeta}{dt} \) for several late years, are the following:

**Hypothesis I.**

<table>
<thead>
<tr>
<th>Year</th>
<th>( \delta \zeta )</th>
<th>( \frac{d\zeta}{dt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1834</td>
<td>- 21 '19</td>
<td>-20 '93</td>
</tr>
<tr>
<td>1840</td>
<td>- 74 '82</td>
<td>-32 '34</td>
</tr>
<tr>
<td>1846</td>
<td>-148 '65</td>
<td>-39 '94</td>
</tr>
</tbody>
</table>

**Hypothesis II.**

<table>
<thead>
<tr>
<th>Year</th>
<th>( \delta \zeta )</th>
<th>( \frac{d\zeta}{dt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1834</td>
<td>- 21 '46</td>
<td>-20 '85</td>
</tr>
<tr>
<td>1840</td>
<td>- 74 '40</td>
<td>-31 '62</td>
</tr>
<tr>
<td>1846</td>
<td>-145 '91</td>
<td>-38 '30</td>
</tr>
</tbody>
</table>
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Hence, by means of the above formulæ, we find the corrections of the tabular radius vector, to be

<table>
<thead>
<tr>
<th>Year</th>
<th>Hypothesis I.</th>
<th>Hypothesis II.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1834</td>
<td>+0.00505</td>
<td>+0.00492</td>
</tr>
<tr>
<td>1840</td>
<td>+0.00722</td>
<td>+0.00696</td>
</tr>
<tr>
<td>1846</td>
<td>+0.00868</td>
<td>+0.00825</td>
</tr>
</tbody>
</table>

59. By far the most important part of these corrections arises from the term 
\(-\frac{1}{2} r \frac{d\hat{c}^2}{ndt}\), and may therefore be immediately deduced from a comparison of the observed angular motion of Uranus with that given by the Tables. In fact, the corrections given by this term alone for the epochs above-mentioned, are

<table>
<thead>
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<th>Year</th>
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<th>Hypothesis II.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1834</td>
<td>+0.00447</td>
<td>+0.00445</td>
</tr>
<tr>
<td>1840</td>
<td>+0.00694</td>
<td>+0.00678</td>
</tr>
<tr>
<td>1846</td>
<td>+0.00853</td>
<td>+0.00818</td>
</tr>
</tbody>
</table>

which, as we see, differ very little from the complete values just found. The correction for 1834, very nearly agrees with that which Mr. Airy has deduced from observation in the Astronomische Nachrichten. The corrections for subsequent years are rather larger than those given by the Greenwich Observations, the results of the second hypothesis being, as in the case of the longitude, nearer the truth than those of the first.

60. I made some attempts, by discussing the observations of latitude, to find approximate values of the longitude of the node and inclination of the disturbing planet, but the results were not satisfactory. The perturbations of the latitude are in fact exceedingly small, and during the comparatively short period of three-fourths of a revolution, are nearly confounded with the effects of a constant alteration in the inclination and the position of the node of Uranus, so that very small errors in the observations may entirely vitiate the result.

61. The perturbations of Saturn produced by the new planet, though small, will still be sensible, and it would be interesting to inquire whether, if they were taken into account, the values of the masses of Jupiter and Uranus found from their action on Saturn would be more consistent with those determined by other means, than they appear to be at present. The reduction of the Greenwich Planetary Observations renders such an inquiry comparatively easy, and it is to be hoped that English astronomers will not be the last to avail themselves of the treasures of observation thus laid open to the world.

St. John's College, Cambridge,
November 12, 1846.

J. C. Adams.
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